Foundations of Classical Electrodynamics[1,2]

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[1] **Keywords**: *electrodynamics, electromagnetism, axiomatics, exterior calculus, classical field theory, (coupling to) gravitation, computer algebra.*– The book is written in *American English* (or what the authors conceive as such). All *footnotes* in the book will be collected at the end of each Part before the references. The image created by *Glatzmaier* (Fig. B.3.4) is in *color*, possibly also Fig. B.5.1 on the aspects of the electromagnetic field. The present format of the book should be changed such as to allow for a 1-line display of longer mathematical formulas.

[2] The book is written in Latex. Our masterfile is **birk.tex**. The different part of the books correspond to the files **partI.tex, partA.tex, partB.tex, partC.tex, partD.tex, partE.tex,** partO.tex. As of today, only a truncated version is available of the outlook chapter on the file **partO.trunc.tex**. The completed version will be handed in later as file partO.tex. – The figures are on separate files. Their names are presently given on the first page of the text of each part of an outprint. For some of the figures we have extra files available with a higher resolution which are, however, not attached to the draft version of the book. Follow up of commands on a Unix system: cd birk, latex birk (2 times), makeindex birk, latex birk, dvips -f birk > birk.ps, gv birk.ps&.

Contents

file birk/partI.tex, with figures [I01cover.eps, I02cover.eps], 2001-06-01

Description of the book

Electric and magnetic phenomena are omnipresent in modern life. Their non-quantum aspects are successfully described by classical electrodynamics (Maxwell's theory). In this book, which is an outgrowth of a physics graduate course, the fundamental structure of classical electrodynamics is presented in the form of six axioms: (1) electric charge conservation, (2) existence of the Lorentz force, (3) magnetic flux conservation, (4) localization of electromagnetic energy-momentum, (5) existence of an electromagnetic spacetime relation, and (6) splitting of the electric current in material and external pieces.

The first four axioms are well-established. For their formulation an arbitrary 4-dimensional differentiable manifold is required which allows for a foliation into 3-dimensional hypersurfaces. The fifth axiom characterizes the environment in which the electromagnetic field propagates, namely *spacetime* with or without gravitation. The relativistic description of such general environments remains a research topic of considerable interest. In particular, it is only in this fifth axiom that the *metric tensor* of spacetime makes its appearance, thus coupling electromagnetism and gravitation. The operational interpretation of the physical notions introduced is stressed throughout. In particular, the electrodynamics of *moving matter* is developed ab initio.

The tool for formulating the theory is the calculus of *exterior differential forms* which is explained in sufficient detail, including the corresponding computer algebra programs.

This book presents a fresh and original exposition of the foundations of classical electrodynamics in the tradition of the so-called metric-free approach. The reader will win a new outlook on the interrelationship and the inner working of Maxwell's equations and their raison d'être.

Book Announcement (old version of 1999):

Electric and magnetic phenomena play an important role in the natural sciences. They are described by means of *classical electrodynamics*, as formulated by Maxwell in 1864. As long as the electromagnetic field is not quantized, Maxwell's equations provide a correct description of electromagnetism.

In this work the foundations of classical electrodynamics are displayed in a consistent axiomatic way. The presentation is based on the simple but far-reaching *axioms* of electric charge conservation, the existence of the Lorentz force, magnetic flux conservation, and the localization of energy-momentum. While the first four axioms above are well-established, they are insufficient to complete the theory. The missing ingredient is a characterization of the environment in which the electromagnetic field propagates by means of *constitutive relations*. This environment is *spacetime* with or without a material medium and with or without gravitation (curvature, perhaps torsion). The relativistic description of such general environments remains a research topic of considerable interest. In particular, it is only in this last axiom that the spacetime metric tensor makes its appearance, thus coupling electromagnetism and gravitation.

The appropriate tool for this theory is the calculus of *exterior differential forms*, which is introduced before the axioms of electrodynamics are formulated. The operational interpretation of the physical notions introduced is stressed throughout.

The book may be used for a course or seminar in theoretical electrodynamics by advanced undergraduates and graduate students in *mathematics, physics, and electrical engeneering.* Its approach to the fundamentals of classical electrodynamics will also be of interest to researchers and instructors as well in the above-mentioned fields.

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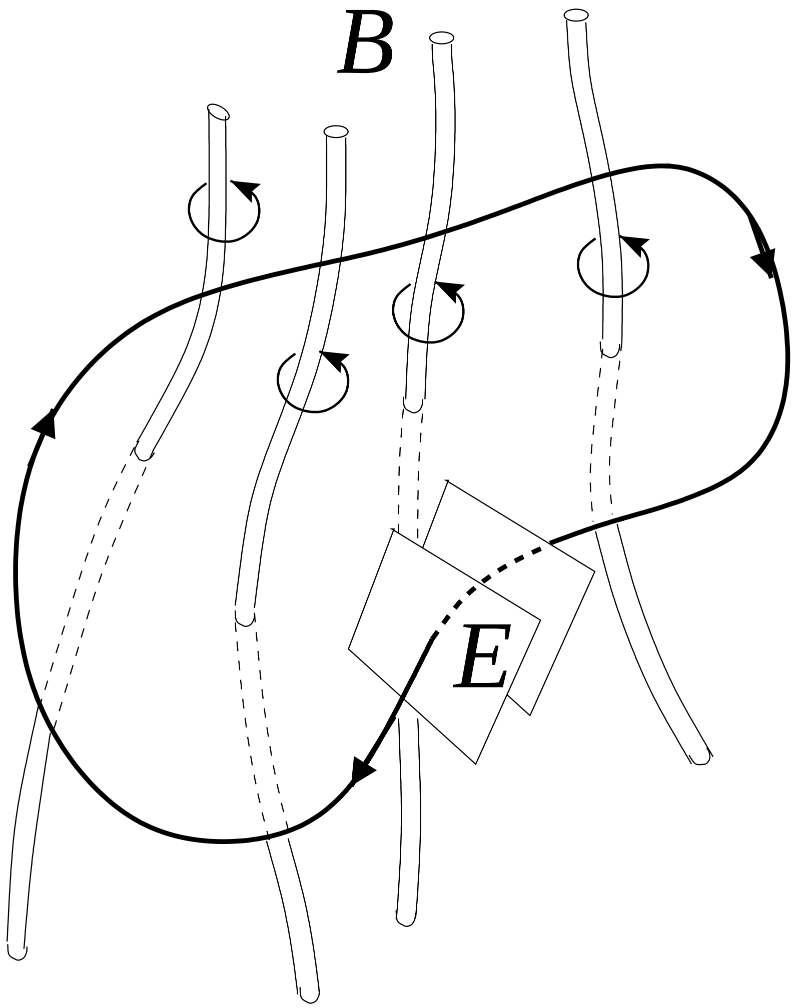

Figure 1: Book cover alternative 1: Faraday's induction law, with drawing program.

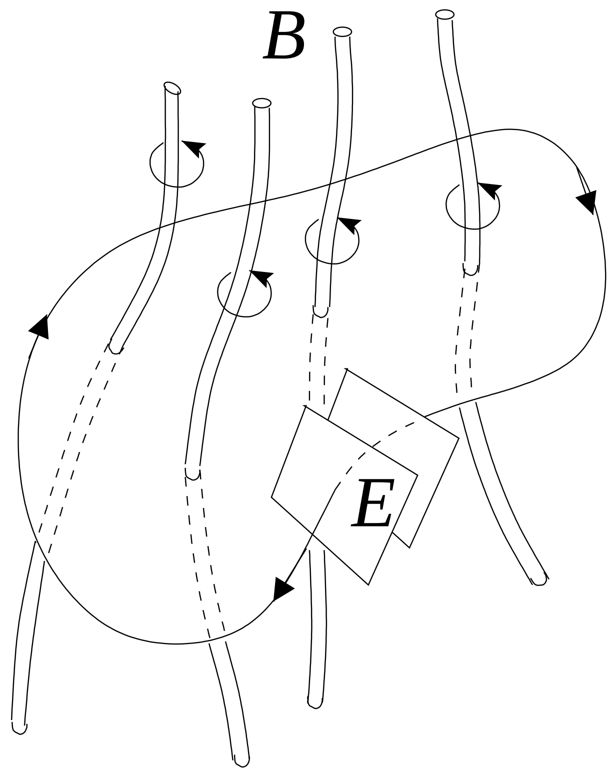

Figure 2: Book cover alternative 2: Faraday's induction law, hand-drawn.

Preface

In this book we will display the fundamental structure underlying classical electrodynamics, i.e., the phenomenological theory of electric and magnetic effects. The book can be used as supplementary reading during a graduate course in theoretical electrodynamics for physics and mathematics students and, perhaps, for some advanced electrical engineering students.

Our approach rests on the metric-free integral formulation of the conservation laws of electrodynamics in the tradition of F. Kottler (1922), E. Cartan (1923), and D. van Dantzig (1934), and we stress, in particular, the axiomatic point of view. In this manner we are led to an understanding of why the Maxwell equations have their specific form. We hope that our book can be seen in the classical tradition of the book by E.J. Post (1962) on the *Formal structure of electromagnetics* and of the chapter on *Charge and magnetic flux* of the encyclopedic article on classical field theories by C. Truesdell and R.A. Toupin (1960), including R.A. Toupin's Bressanone lectures (1965); for the exact references see the end of the Introduction on page @@@.

The manner in which electrodynamics is conventionally presented in physics courses à la R. Feynman (1962) or J.D. Jackson (1999) is distinctively different, since it is based on a flat spacetime manifold, i.e., on the (rigid) Poincaré group, and on H.A. Lorentz's approach (1916) to Maxwell's theory by means of his theory of electrons. We believe that the approach of this book is appropriate and, in our opinion, even superior for a good understanding of the structure of electrodynamics as a *classical field theory*. In particular, if gravity cannot be neglected, our framework allows for a smooth and trivial transition to the curved (and contorted) spacetime of general relativistic field theories.

Mathematically, integrands in the conservation laws are represented by exterior differential forms. Therefore exterior calculus is the appropriate language in which electrodynamics should be spelled out. Accordingly, we will exclusively use this formalism (even in our computer algebra programs which we will introduce in Sec. A.1.12.). In an introductory Part A, and later in Part C, we try to motivate and to supply the necessary mathematical framework. Readers who are familiar with this formalism may want to skip these parts. They could start right away with the physics in Part B and then turn to Part D and Part E.

In Part B four axioms of classical electrodynamics are formulated and the consequences derived. This general framework has to be completed by a specific *electromagnetic spacetime relation* as fifth axiom. This will be done in Part D. The Maxwell-Lorentz approach is then recovered under specific conditions. In Part E, we will apply electrodynamics to moving continua, inter alia, which requires a sixth axiom on the formulation of electrodynamics inside matter.

These notes grew out of a scientific collaboration with the late *Dermott McCrea* (University College Dublin). Mainly in Part A and Part C Dermott's handwriting can still be seen in numerous places. There are also some contributions to 'our' mathematics from *Wojtek Kopczyński* (Warsaw University). At Cologne University in the summer term of 1991, *Martin Zirnbauer* started to teach the theoretical electrodynamics course by using the calculus of exterior differential forms, and he wrote up successively improved notes to his course. One of the authors (FWH) also taught this course three times, partly based on M. Zirnbauer's notes. This influenced our way of presenting electrodynamics (and, we believe, also his way). We are very grateful to M. Zirnbauer for many discussions.

There are many colleagues and friends who helped us in critically reading parts of our book and who made suggestions for improvement or who communicated to us their own ideas on electrodynamics. We are very grateful to all of them: Carl Brans (New Orleans), David Hartley (Adelaide), Yakov Itin (Jerusalem), Martin Janssen (Cologne), Gerry Kaiser (Glen Allen, Virginia), R.M. Kiehn (formerly Houston), Attay Kovetz (Tel Aviv), Claus Lämmerzahl (Konstanz/Düsseldorf), Bahram Mashhoon (Columbia, Missouri), Eckehard Mielke (Mexico City), E. Jan Post (Los Angeles), Dirk Pützfeld (Cologne), Guillermo Rubilar (Cologne), Yasha Shnir (Cologne), Andrzej Trautman (Warsaw), Wolfgang Weller (Leipzig), and others.

We are very obliged to Uwe Essmann (Stuttgart) and to Gary Glatzmaier (Santa Cruz, California) for providing beautiful and instructive images. We are equally grateful to Peter Scherer (Cologne) for his permission to reprint his three comics on computer algebra.

Please let us know critical remarks to our approach or the discovery of mistakes by surface or electronic mail (hehl@thp.uni-koeln.de, yo@thp.uni-koeln.de).

This project has been supported by the Alexander von Humboldt-Foundation (Bonn), the German Academic Exchange Service DAAD, and the Volkswagen-Foundation (Hanover). We are very grateful for the unbureaucratic help of these institutions.

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Introduction

Five plus one axioms

In this book we will display the structure underlying classical electrodynamics. For this purpose we will formulate six axioms: Conservation of electric *charge* (first axiom), existence of the Lorentz *force* (second axiom), conservation of magnetic *flux* (third axiom), local *energy-momentum* distribution (fourth axiom), existence of an electromagnetic *spacetime* relation (fifth axiom), and, eventually, the splitting of the electric current in material and external pieces (sixth axiom).

The axioms of the conservation of electric charge and magnetic flux will be formulated as integral laws, whereas the axiom for the Lorentz force is represented by a *local* expression basically defining the electromagnetic field strength $F = (E, B)$ as force per unit charge and thereby linking electrodynamics to mechanics; here E is the electric and B the magnetic field strength. Also the energy-momentum distribution is specified as a local law. The Maxwell-Lorentz spacetime relation, that we will use, is, as an axiom, not so unquestionable as the first four axioms and non-local and non-linear alternatives will be mentioned.

We want to stress the fundamental nature of the *first* axiom. Electric charge conservation is experimentally firmly established. It is valid for single elementary particle processes (like the β-decay, $n \to p^+ + e^- + \overline{\nu}$, for instance, with n as neutron, p as proton, e as electron, and $\overline{\nu}$ as electron anti-neutrino). In other words, it is a *microscopic* law valid without any known exception.

Accordingly, it is basic to electrodynamics to assume a new type of entity, called *electric charge*, carrying positive or negative sign, with its own physical dimension, independent of the classical fundamental variables mass, length, and time. Furthermore, electric charge is conserved. In an age in which single electrons and (anti-)protons are *counted* and caught in traps, this law is so deeply ingrained in our thinking that its explicit formulation as a fundamental law (and not only as a consequence of Maxwell's equations) is often forgotten. We will show that this first axiom yields the inhomogeneous Maxwell equation together with a definition of the electromagnetic excitation $H = (\mathcal{H}, \mathcal{D})$; here $\mathcal{H}$ is the magnetic excitation ('magnetic field') and $\mathcal{D}$ the electric excitation ('dielectric displacement'). The excitation H is a microscopic field of an analogous quality as the field strength F. There exist operational definitions of the excitations $\mathcal{D}$ and $\mathcal{H}$ (via Maxwellian double plates or a compensating superconducting wire, respectively).

The *second* axiom of the Lorentz force, as mentioned above, leads to the notion of the field strength and is thereby exhausted. Thus we need further axioms. The only conservation law which can be naturally formulated in terms of the field strength, is the conservation of magnetic flux (lines). This *third* axiom has the homogeneous Maxwell equation as a consequence, that is, Faraday's induction law and the vanishing divergence of the magnetic field strength. Magnetic monopoles are alien to the structure of the axiomatics we are using.

Moreover, with the help of these first three axioms, we are led, not completely uniquely, however, to the electromagnetic *energy-momentum* current (fourth axiom), subsuming the energy and momentum densities of the electromagnetic field and their corresponding fluxes, and to the *action* of the electromagnetic field. In this way, the basic structure of electrodynamics is set up including the complete set of Maxwell's equations. For making this set of electrodynamic equations well–determined, we still have to add the fifth axiom.

Let us come back to the magnetic monopoles. In our axiomatic framework, a clear *asymmetry* is built in between electricity and magnetism in the sense of Oersted and Ampère that magnetic effects are created by moving electric charges. This asymmetry is characteristic for and intrinsic to Maxwell's theory. Therefore the conservation of magnetic *flux* and not that of magnetic charge is postulated as third axiom. If one speculated on a possible violation of the third axiom, i.e., introduced elementary magnetic charges, so-called magnetic monopoles, then — in our framework — there would be no reason to believe any longer in electric charge conservation either. If the third axiom is violated, why should then the first axiom, electric charge conservation, be untouchable? In other words, if ever a *magnetic* monopole[1] is found, our axiomatics has to be given up. Or, to formulate it more positively: Not long ago, He [20], Abbott et al. [1], and Kalbfleisch et al. [29] determined experimentally new improved limits for the non-existence of (Abelian or Dirac) magnetic monopoles. This increasing accuracy in the exclusion of magnetic monopoles speaks in favor of the axiomatic approach in Part B.

Topological approach

Since the notion of metric is a complicated one, which requires measurements with clocks and scales, generally with rigid *bodies, which themselves are systems of great complexity, it seems undesirable to take metric as fundamental, particularly for phenomena which are simpler and actually independent of it.*

E. Whittaker (1953)

The distinctive feature of this type of axiomatic approach is that one only needs minimal assumptions about the structure of the spacetime in which these axioms are formulated. For the first four axioms, a 4-dimensional differentiable manifold is required which allows for a *foliation* into 3-dimensional hypersurfaces. Thus no *connection* and no *metric* are explicitly introduced in Parts A and B.

This minimalistic *topological type* of approach may appear contrived at a first look. We should recognize, however, that the metric of spacetime, in the framework of general relativity theory, represents the gravitational potential and, similarly, the connection of spacetime (in the viable Einstein-Cartan theory of gravity, e.g.) is intricately linked to gravitational properties of matter. We know that we really live in a curved and, perhaps, contorted spacetime. Consequently our desire should be to formulate the foundations of electrodynamics such that the

[1] Our arguments refer only to Abelian gauge theory. In *non*-Abelian gauge theories the situation is totally different. There monopoles are a must.

metric and the connection don't intervene or intervene only in the least possible way. When we know that the gravitational field permeates all our laboratories in which we make experiments with electricity, all the more we should take care that this ever present field doesn't enter the formulation of the first principles of electrodynamics. In other words, a *clear separation* between pure electrodynamic effects and gravitational effects is desirable and can, indeed, be achieved by means of the axiomatic approach to be presented in Part B. Eventually, in the spacetime relation, see Part D, the metric does enter.

The power of the topological approach is also clearly indicated by its ability to describe the *phenomenology* (at low frequencies and large distances) of the quantum Hall effect successfully (not, however, its quantization, of course). Insofar as the macroscopic aspects of the quantum Hall effect can be approximately understood in terms of a 2-dimensional electron gas, we can start with (1+2)-dimensional electrodynamics, the formulation of which is straightforward in our axiomatics. It is then a sheer finger exercise to show that in this specific case of 1+2 dimensions there exists a *linear* constitutive law that doesn't require a metric. As a consequence the action is metric-free, too. Thus the formulation of the quantum Hall effect by means of a *topological* (Chern-Simons) Lagrangian is imminent in our way of looking at electrodynamics.

Electromagnetic spacetime relation as fifth axiom

Let us now turn to that domain where the metric does enter the 4-dimensional electrodynamical formalism. When the Maxwellian structure, including the Lorentz force and the action, is set up, it does not represent a concrete physical theory yet. What is missing is the electromagnetic spacetime relation linking the excitation to the field strength, i.e., $\mathcal{D} = \mathcal{D}(E, B)$, $\mathcal{H} = \mathcal{H}(B, E)$, or, written 4-dimensionally, $H = H(F)$.

Trying the simplest, we assume *linearity* between excitation H and field strength F, that is, $H = \boldsymbol{\kappa}(F)$, with the linear operator $\boldsymbol{\kappa}$. Together with two more "technical" assumptions, namely that $\boldsymbol{\kappa}$ fulfills *closure* and *symmetry* – these properties will be discussed in detail in Part D – we will be able to *derive* the metric of spacetime from $H = \boldsymbol{\kappa}(F)$ up to an arbitrary (conformal) factor. Accordingly, the lightcone structure of spacetime is a consequence of a linear electromagnetic spacetime relation with the additional properties of closure and symmetry. In this sense, the lightcones are derived from electrodynamics:

Electrodynamics doesn't live in a preformed rigid Minkowskian spacetime. It rather has an arbitrary $(1+3)$-dimensional spacetime manifold as habitat which, as soon as a linear spacetime relation with closure and symmetry is supplied, will be equipped with local lightcones everywhere. With the lightcones it is possible to define the Hodge star operator * that can map p-forms to $(4-p)$-forms and, in particular, H to F according to $H \sim {}^\star F$.

Thus, in the end, that property of spacetime which describes its local 'constitutive' structure, namely the metric, enters the formalism of electrodynamics and makes it into a complete theory. One merit of our approach is that it doesn't matter whether it is the rigid, i.e., flat *Minkowskian* metric as in special relativity, or a 'flexible' *Riemannian* metric field which changes from point to point according to Einstein's field equation as in general relativitistic gravity theory. In this way, the traditional discussion of how to translate electrodynamics from special to general relativity loses its sense: The Maxwell equations remain the same, i.e., the exterior derivatives (the 'commas' in coordinate language) are kept and are *not* substituted by something 'covariant' (the 'semicolons'), and the spacetime relation $H = \lambda\, {}^\star F$ looks the same (λ is a suitable factor). However, the Hodge star 'feels' the difference in referring either to a constant or to a spacetime dependent metric, respectively, see [46, 21].

Our formalism can accomodate generalizations of classical electrodynamics simply by suitably modifying the fifth axiom, while keeping the first four axioms as indispensable. The Heisenberg-Euler and the Born-Infeld electrodynamics are prime examples of such possible modifications. The spacetime relation becomes a *non*-linear, but still local expression.

Electrodynamics in matter and the sixth axiom

Eventually, we have to face the problem of formulating electrodynamics inside matter. We codify our corresponding approach in the sixth axiom. The total electric current, entering as source the inhomogeneous Maxwell equation, is split into a conserved piece carried by *matter* ('bound charge') and into an *external* manipulable piece ('free charge'). In this way, following Truesdell & Toupin [57], see also the textbook of Kovetz [31], we can develop a consistent theory of electrodynamics in matter. For simple cases, we can amend the axioms by a *linear constitutive law*. We believe that the conventional theory of electrodynamics inside matter needs to be redesigned.

In order to demonstrate the effectiveness of our formalism, we apply it to the electrodynamics of moving matter, thereby coming back to the post-Maxwellian time of the 1880's when the relativistic version of Maxwell's theory had won momentum. In this context, we discuss and analyse the experiments of Röntgen-Eichenwald, Wilson & Wilson, and Walker & Walker.

List of axioms

1. Conservation of electric charge: (B.1.17).
2. Lorentz force density: (B.2.6).
3. Conservation of magnetic flux: (B.3.1).
4. Localization of energy-momentum: (B.5.8).
5. Maxwell-Lorentz spacetime relation: (D.5.7).
6. Splitting of the electric current in a conserved matter piece and an external piece: (E.3.1) and (E.3.2).

A reminder: Electrodynamics in 3-dimensional Euclidean vector calculus

Before we will start to develop electrodynamics in 4-dimensional spacetime in the framework of the calculus of exterior differential forms, it may be useful to remind ourselves of electrodynamics in terms of conventional 3-dimensional Euclidean vector calculus. We begin with the laws obeyed by electric charge and current.

If $\vec{\mathcal{D}} = (\mathcal{D}_x, \mathcal{D}_y, \mathcal{D}_z)$ denotes the *electric excitation* field (historically 'dielectric displacement') and ρ the electric charge density, then the integral version of the *Gauss law*, 'flux of $\mathcal{D}$ through any closed surface' equals 'net charge inside', reads

$$\int_{\partial V} \vec{\mathcal{D}} \cdot \vec{df} = \int_{V} \rho \, dV \,, \tag{1}$$
M1

with $\vec{df}$ as area and dV as volume element. The *Oersted-Ampère law* with the *magnetic excitation* field $\vec{\mathcal{H}} = (\mathcal{H}_x,\, \mathcal{H}_y, \mathcal{H}_z)$ (historically 'magnetic field') and the electric current density $\vec{j} = (j_x, j_y, j_z)$ is a bit more involved because of the presence of the Maxwellian electric excitation current: The 'circulation of $\vec{\mathcal{H}}$ around any closed contour' equals '$\frac{d}{dt}\Big($flux of $\vec{\mathcal{D}}$ through surface spanned by contour$\Big)$' plus 'flux of $\vec{j}$ through surface' ($t =$time):

$$\oint_{\partial S} \vec{\mathcal{H}} \cdot \vec{dr} = \frac{d}{dt}\left(\int_{S} \vec{\mathcal{D}} \cdot \vec{df}\right) + \int_{S} \vec{j} \cdot \vec{df} \,. \tag{2}$$
M2

Here $\vec{dr}$ is the vectorial line element. Here the dot $\cdot$ always denotes the 3-dimensional metric dependent scalar product, S denotes a 2-dimensional spatial surface, V a 3-dimensional spatial volume, and ∂S and ∂V the respective boundaries. Later we will recognize that both, (1) and (2), can be derived from the *charge conservation law.*

The *homogeneous* Maxwell equations are formulated in terms of the electric field strength $\vec{E} = (E_x, E_y, E_z)$ and the magnetic field strength $\vec{B} = (B_x, B_y, B_z)$. They are defined operationally via the expression of the Lorentz force $\vec{F}$. An electrically charged particle with charge q and velocity $\vec{v}$ experiences the force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \,. \tag{3}$$
M3

Here the cross $\times$ denotes the 3-dimensional vector product. Then *Faraday's induction law* in its integral version, namely 'circulation of $\vec{E}$ around any closed contour' equals 'minus $\frac{d}{dt}\Big($flux of $\vec{B}$ through surface spanned by contour$\Big)$' reads:

$$\oint_{\partial S} \vec{E} \cdot d\vec{r} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{f}\right) . \tag{4}$$ M4

Note the minus sign on its right hand side which is chosen according to the Lenz rule (following from energy conservation). Eventually, the 'flux of $\vec{B}$ through any closed surface' equals 'zero', that is,

$$\int_{\partial V} \vec{B} \cdot d\vec{f} = 0 . \tag{5}$$ M5

Also (4) and (5) are inherently related. Later we will find the law of *magnetic flux conservation* – and (4) and (5) just turn out to be consequences of it.

Applying the Gauss and the Stokes theorems, the integral form of the Maxwell equations (1, 2) and (4, 5) can be transformed into their differential versions:

$$\operatorname{div} \vec{\mathcal{D}} = \rho , \qquad \operatorname{curl} \vec{\mathcal{H}} - \dot{\vec{\mathcal{D}}} = \vec{j} , \tag{6}$$ M6

$$\operatorname{div} \vec{B} = 0 , \qquad \operatorname{curl} \vec{E} + \dot{\vec{B}} = 0 . \tag{7}$$ M7

Additionally, we have to specify the spacetime relations $\vec{\mathcal{D}} = \varepsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{\mathcal{H}}$ and possibly the constitutive laws.

This formulation of electrodynamics by means of 3-dimensional Euclidean vector calculus represents only a preliminary version, since the 3-dimensional metric enters the scalar and the vector products and, in particular, the differential operators div $\equiv \vec{\nabla}\cdot$ and curl $\equiv \vec{\nabla}\times$, with $\vec{\nabla}$ as the nabla operator. In the Gauss law (1) or $(6)_1$, for instance, only *counting procedures* enter, namely counting of elementary charges inside V – taking also care of their sign, of course – and counting of flux lines piercing through a closed surface ∂V. No length or time measurements and thus no metric is involved in such type of processes, as will be described in more detail below. Since similar arguments apply also to (5) or $(7)_1$, respectively, it should be possible to remove the metric from the Maxwell equations altogether.

On the literature

Basically not too much is new in our book. Probably Part D is the most original one. Most of the material can be found somewhere in the literature. What we do claim, however, is some originality in the *completeness* and in the appropriate *arrangement* of the material which is fundamental to the structure electrodynamics is based on. Moreover, we try to stress the *phenomena* underlying the axioms chosen and the *operational interpretation* of the quantities introduced. The *explicit derivation* in Part D of the metric of spacetime from pre-metric electrodynamics by means of linearity, reciprocity, and symmetry, though considered earlier mainly by Toupin [55], Schönberg [48], and Jadczyk [26], is new and rests on recent results of Fukui, Gross, Rubilar, and the authors [38, 22, 37, 19, 47].

Our *main sources* are the works of Post [41, 42, 43, 44, 45], of Truesdell & Toupin [57], and of Toupin [55]. Historically, the metric-free approach to electrodynamics, based on integral conservation laws, was pioneered by Kottler [30], E.Cartan [9], and van Dantzig [58], also the article of Einstein [15] and the books of Mie [35], Weyl [59], and Sommerfeld [52] should be consulted on these matters, see also the recent textbook of Kovetz [31]. A description of the corresponding historical development, with references to the original papers, can be found in Whittaker [60] and, up to about 1900, in the penetrating account of Darrigol [11]. The driving forces and the results of Maxwell in his research on electrodynamics are vividly presented in Everitt's [16] concise biography of Maxwell.

In our book, we will consistently use exterior calculus[2], including de Rham's *odd* (or twisted) differential forms. *Textbooks* on electrodynamics using exterior calculus are scarce. We know only of Ingarden & Jamiołkowski [24], in German of Meetz & Engl [34] and Zirnbauer [61] and, in Polish, of Jancewicz [28], see also [27]. However, as a mathematical physics' discipline, corresponding presentations can be found in Bamberg & Sternberg [4], in Thirring [54], and, as a short sketch, in Piron [39], see also [5, 40]. Bamberg & Sternberg are particular easy to follow and present electrodynamics in a very transparent way. That electrodynamics in the framework of exterior calculus is also in the scope of electrical engineers, can be seen from Deschamps [13] and Baldomir & Hammond [3], e.g..

Presentations of exterior calculus, partly together with applications in physics and electrodynamics, were given, amongst many others, by Burke [8], Choquet-Bruhat et al. [10], Edelen [14], and Frankel [18]. For differential geometry we refer to the classics of de Rham [12] and Schouten [49, 50] and to Trautman [56].

What else did influence the writing of our notes? The *axiomatics* of Bopp [7] is different but related to ours. In the more microphysical axiomatic attempt of Lämmerzahl et al., Maxwell's equations [32] (and the Dirac equation [2]) are deduced from direct experience with electromagnetic (and matter) waves, inter alia. The clear separation of *differential*, *affine*, and *metric* structures of spacetime is nowhere more pronounced than in Schrödinger's [51] 'Space-time structure'. A further presentation of electrodynamics in this spirit, somewhat similar to that of Post, has been given by Stachel [53]. Our (1+3)-*decomposition* of spacetime is based on the paper by Mielke & Wallner [36]. Recently, Hirst [23] has shown, mainly based on experience with neutron scattering on magnetic structures in solids, that *magnetization* $\mathcal{M}$ is a *microscopic* quantity. This is in accord with

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[2]Baylis [6] also advocates a geometric approach, using Clifford algebras. In such a framework, however, at least in the way Baylis does it, the metric of spacetime is introduced right from the beginning. In this sense, Baylis's Clifford algebra approach is complementary to our metric-free electrodynamics.

our axiomatics which yields the magnetic excitation $\mathcal{H}$ as microscopic quantity, quite analogously to the field strength B, whereas in conventional texts $\mathcal{M}$ is only defined as a macroscopic *average* over microscopically fluctuating magnetic fields. Clearly, with $\mathcal{H}$, also the electric excitation $\mathcal{D}$, i.e. the electromagnetic excitation $H = \{\mathcal{H}, \mathcal{D}\}$ altogether, ought to be a microscopic field.

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Part A

Mathematics: Some exterior calculus

Why exterior differential forms?

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In this Part A and later, in Part C, we shall be concerned with assembling the geometric concepts in the language of differential forms that are needed to formulate a classical field theory like *electrodynamics* and/or the theory of *gravitation*. The basic geometric structure underlying such a theory is that of a spacetime continuum or, in mathematical terms, a 4-dimensional *differentiable manifold* X_4. The characteristics of the gravitational field will be determined by the nature of the additional geometric structures that are superimposed on this 'bare manifold' X_4. For instance, in Einstein's general relativity theory (GR), the manifold is endowed with a *metric* together with a torsion-free, metric-compatible connection: It is a 4-dimensional Riemannian spacetime V_4. The meaning of these terms will be explained in detail in what follows.

In Maxwell's theory of electrodynamics, under most circumstances, gravity can safely be neglected. Then the Riemannian spacetime becomes *flat*, i.e. its curvature vanishes, and we have the (rigid) Minkowskian spacetime M_4 of special relativity theory (SR). Its spatial part is the ordinary 3-dimensional Euclidean space R_3. However, and this is one of the messages of the book, for the fundamental axioms of electrodynamics we don't need to take into account the metric structure of spacetime and, even more so, we should not take it into account. This helps to keep electrodynamical structures cleanly separated from the gravitational ones.

This separation is particularly decisive for a proper understanding of the emergence of the *lightcone*. On the one side, by its very definition, it is an electrodynamical concept in that it determines the front of a propagating electromagnetic disturbance; on the other hand it constitutes the main (conformally invariant) part of the metric tensor of spacetime and is as such part of the gravitational potential of GR. This complicated interrelationship we will try to untangle in Part D.

A central role in the formulation of classical electrodynamics adopted in the present work will be played by the conservation laws of electric charge and magnetic flux. We will start from their integral formulation. Accordingly, there is a necessity for an adequate understanding of the concepts involved when one writes down an *integral* over some domain on a differentiable manifold.

Specifically, in the Euclidean space R_3 in Cartesian coordinates, one encounters integrals like the electric tension (voltage)

$$\int_C (E_x\,dx + E_y\,dy + E_z\,dz) \qquad (8)$$

intC

evaluated along a (1-dimensional) curve C, the magnetic flux

$$\int_S (B_x\,dy\,dz + B_y\,dz\,dx + B_z\,dx\,dy) \qquad (9)$$

intflux

over a (2-dimensional) surface S, and the (total) charge

$$\int_V \rho \, dx \, dy \, dz \tag{10}$$

intcharge

integrated over a (3-dimensional) volume V.

The fundamental result of classical integral calculus is Stokes's theorem which relates an integral over the boundary of a region to one taken over the region itself. Familiar examples of this theorem are provided by the expressions

$$\int_{\partial S} (E_x \, dx + E_y \, dy + E_z \, dz) = \int_S \left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) dy \, dz + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) dz \, dx + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) dx \, dy \right], \tag{11}$$

stokes1

and

$$\int_{\partial V} (B_x \, dy \, dz + B_y \, dz \, dx + B_z \, dx \, dy) = \int_V \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) dx \, dy \, dz \, , \tag{12}$$

stokes2

where ∂S and ∂V are the boundaries of S and V, respectively. The right hand sides of these equations correspond to $\int_S \mathrm{curl}\vec{E} \cdot d\vec{f}$ and $\int_V \mathrm{div}\vec{B} \, dV$, respectively.

Consider the integral

$$\int \rho(x, y, z) \, dx \, dy \, dz \tag{13}$$

intrho

and make a change of variables:

$$x = x(u, v, w) \, , \qquad y = y(u, v, w) \, , \qquad z = z(u, v, w) \, . \tag{14}$$

uvw

For simplicity and only for the present purpose, let us suppose that the Jacobian determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad (15)$$ jacobian

is *positive*. We obtain

$$\int \rho(x,y,z)\,dx\,dy\,dz = \int \rho[x(u,v,w),y(u,v,w),z(u,v,w)]\,\frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw\,. \qquad (16)$$ intrho2

This suggests that we should write

$$dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}\,du\,dv\,dw\,. \qquad (17)$$ jacobian2

If we set $x = y$ or $x = z$ or $y = z$ the determinant has equal rows and hence vanishes. Also, an odd permutation of x, y, z changes the sign of the determinant while an even permutation leaves it unchanged. Hence, we have

$$dx\,dx = 0\,, \qquad dy\,dy = 0\,, \qquad dz\,dz = 0\,, \qquad (18)$$

$$dx\,dy\,dz = dy\,dz\,dx = dz\,dx\,dy = -\,dy\,dx\,dz = -dx\,dz\,dy = -dz\,dy\,dx\,. \qquad (19)$$

It is this *alternating algebraic structure of integrands* that gave rise to the development of exterior algebra and calculus which is becoming more and more recognized as a powerful tool in mathematical physics. In general, an exterior p-form will be an expression

$$\boldsymbol{\omega} = \frac{1}{p!}\,\omega_{i_1,\ldots i_p}\,dx^{i_1}\cdots dx^{i_p}\,, \qquad (20)$$ pform

where the components $\omega_{i_1,\dots i_p}$ are completely antisymmetric in the indices and $i_m = 1, 2, 3$. Furthermore, summation from 1 to 3 is understood over repeated indices. Then, when translating (12) in exterior form calculus, we recognize B as a 2-form

$$\int\limits_{\partial V} B = \int\limits_{\partial V} \frac{1}{2!}\, B_{[ij]}\, dx^i dx^j = \int\limits_{V} \frac{1}{3!}\, \partial_{[k} B_{ij]}\, dx^k dx^i dx^j = \int\limits_{V} dB\,. \tag{21}$$

Bform

Note that $_{[ij]} := (\ _{ij} - \ _{ji})/2$, similarly $_{(ij)} := (\ _{ij} + \ _{ji})/2$, etc. Accordingly, in the E_3, we have the *magnetic* field B as *2-form* and, as a look at (8) will show, the *electric* field as *1-form*; and the *charge* ρ in (10) turns out to be a *3-form.*

In the 4-dimensional *Minkowski* space M_4, the electric current J, like ρ in the E_3, is represented by a 3-form. Since the action functional of the electromagnetic field is defined in terms of a 4-dimensional integral, the integrand, the Lagrangian L, is a 4-form. The coupling term in L of the current J to the potential A, namely $\sim J \wedge A$, identifies A as 1-form. In the inhomogeneous Maxwell equation $J = dH$, the 3-form character of J attributes to the excitation H a 2-form. If, eventually, we execute a gauge transformation $A \to A + df$, $F \to F$, we meet a 0-form f. Consequently, a (gauge) field theory, starting from a conserved current 3-form, here the electric current J, generates in a straightforward way forms of all ranks $p \leq 4$.

We know from classical calculus that if the Jacobian determinant (15) above has negative values, i.e. the two coordinate systems do not have the same orientation, then, in equations (16) and (17), the Jacobian determinant must be replaced by its absolute value. In particular, instead of (17), we get the general formula

$$dx\, dy\, dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw\,. \tag{22}$$

jacobian3

This behavior under a change of coordinates is typical of what is known as a *density*. We shall see that, if we wish to drop the requirement that all our coordinate systems should have the same orientation, then densities become important and these latter are closely related to the *twisted differential forms* which one has to introduce – besides the ordinary differential forms – since the electric current, e.g., is of such a twisted type.

In Chapter A.1, we first consider a vector space and its dual and study the algebraic aspect of tensors and of geometrical quantities of a more general type. Then, we turn our attention to exterior forms and their algebra and to a corresponding computer algebra program.

Since the tangent space at every point of a differentiable manifold is a linear vector space, we can associate an exterior algebra with each point and define differentiable *fields* of exterior forms or, more concisely, *differential forms* on the manifold. This is done in Chapter A.2, while Chapter A.3 deals then with integration on a manifold.

It is important to note that in this Part A, we are dealing with the 'bare manifold'. Linear connection and metric will be introduced not before Part C, after the basic axiomatics of electrodynamics will have been laid down earlier in Part B.

A.1

Algebra

A.1.1 A real vector space and its dual

Our considerations are based on an n-dimensional real vector space V. One-forms are the elements of the dual vector space V^* defined as linear maps of the vector space V into the real numbers. The dual bases of V and V^* transform reciprocally to each other with respect to the action of the linear group. The vectors and 1-forms can be alternatively defined by their components with a specified transformation law.

Let V be an n-dimensional real vector space. We can depict a vector $v \in V$ by an arrow. If we multiply v by a factor f, the vector has an f-fold size, see Fig. A.1.1. Vectors are added according to the parallelogram rule. A linear map $\omega : V \to \mathbb{R}$ is called a *1-form* on V. The set of all 1-forms on V can be given a structure of a vector space by defining the sum of two arbitrary 1-forms ω and φ,

$$(\omega + \varphi)(v) = \omega(v) + \varphi(v)\,, \qquad v \in V, \tag{A.1.1}$$ sumforms

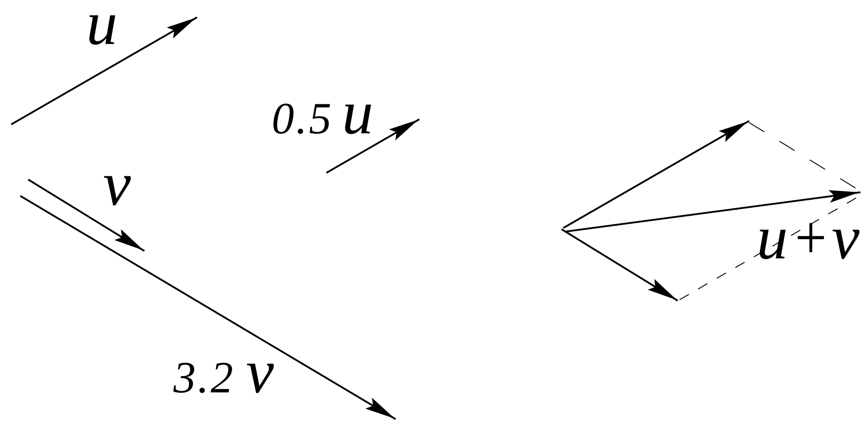

Figure A.1.1: Vectors as arrows, their multiplication by a factor, their addition (by the parallelogram rule).

and the product of ω by a real number $\lambda \in \mathbb{R}$,

$$(\lambda\omega)(v) = \lambda\,(\omega(v))\,. \qquad v \in V. \tag{A.1.2}$$ prodforms

This vector space is denoted V^* and called *dual* of V. The dimension of V^* is equal to the dimension of V. The identification $V^{**} = V$ holds for finite dimensional spaces.

In accordance with (A.1.1), (A.1.2), a 1-form can be represented by a pair of ordered hyperplanes, see Fig. A.1.2. The nearer the hyperplanes are to each other, the stronger is the 1-form. In Fig. A.1.3, the action of a 1-form on a vector is depicted.

Denote by $e_\alpha = (e_1, \ldots, e_n)$ a (vector) *basis* in V. An arbitrary vector v can be decomposed with respect to such a basis: $v = v^\alpha\, e_\alpha$. Summation from 1 to n is understood over repeated indices (Einstein's summation convention). The n real numbers v^α, $\alpha = 1, \ldots, n$, are called components with respect to the given basis. With a basis e_α of V we can associate its dual 1-form, or covector basis, the so-called *cobasis* $\vartheta^\alpha = (\vartheta^1, \ldots, \vartheta^n)$ of V^*. It is determined by the relation

$$\vartheta^\alpha(e_\beta) = \delta^\alpha_\beta\,. \tag{A.1.3}$$ dualbasis

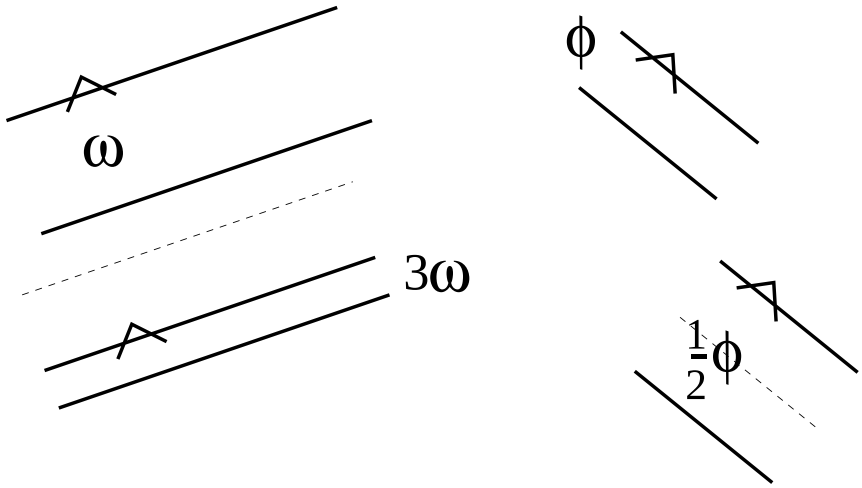

Figure A.1.2: One-forms as two parallel hyperplanes (straight lines in $n = 2$) with a direction, their multiplication by a factor.

Here δ^α_β is the Kronecker symbol with $\delta^\alpha_\beta = 1$ for $\alpha = \beta$ and $\delta^\alpha_\beta = 0$ for $\alpha \neq \beta$.

The components ω_α of a 1-form ω with respect to the cobasis ϑ^α are then given by

$$\omega = \omega_\alpha \, \vartheta^\alpha \qquad \Longrightarrow \qquad \omega_\alpha = \omega(e_\alpha)\,. \tag{A.1.4}$$ formcomps

A transformation from a basis e_α of V to another one ('alpha-prime' basis) $e_{\alpha'} = (e'_1, \ldots, e'_n)$ is described by a matrix $\mathrm{L} := \left(L_{\alpha'}{}^\alpha\right) \in GL(n, \mathbb{R})$ (general linear real n-dimensional group):

$$e_{\alpha'} = L_{\alpha'}{}^\alpha \, e_\alpha\,. \tag{A.1.5}$$ basetrafo

The corresponding cobases are thus connected by

$$\vartheta^{\alpha'} = L_\alpha{}^{\alpha'} \, \vartheta^\alpha\,, \tag{A.1.6}$$ dualtrafo

where $\left(L_\alpha{}^{\alpha'}\right)$ is the inverse matrix to $\left(L_{\alpha'}{}^\alpha\right)$, i.e., $L_\alpha{}^{\alpha'} \, L_{\alpha'}{}^\beta = \delta^\beta_\alpha$. Symbolically, we may also write $e' = \mathrm{L}\, e$ and $\vartheta' = \left(\mathrm{L}^T\right)^{-1} \vartheta$. Here T denotes the transpose of the matrix L.

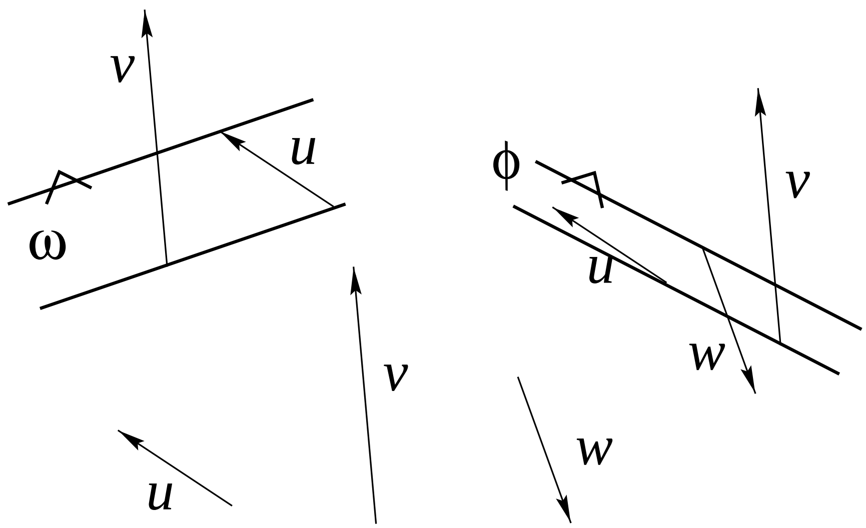

Figure A.1.3: A 1-form acts on a vector. Here we have $\omega(u) = 1$, $\omega(v) \approx 2.3$; $\phi(u) \approx 0.3$, $\phi(v) \approx 4.4$; $\phi(w) \approx -2.1$. A 1-form can be understood as a machine: You input a vector and the output is a number which can be read off from our images.

Consequently, one can view a vector $v \in V$ as n ordered numbers v^α which transform under a change (A.1.5) of the basis as

$$v^{\alpha'} = L_\alpha{}^{\alpha'} v^\alpha \,, \qquad \text{(A.1.7)}$$ vectrafo

whereas a 1-form $\omega \in V^*$ is described by its components ω_α with the transformation law

$$\omega_{\alpha'} = L_{\alpha'}{}^{\alpha} \omega_\alpha \,. \qquad \text{(A.1.8)}$$ omtrafo

The similarity of (A.1.7) to (A.1.6) and of (A.1.8) to (A.1.5) and the fact that the two matrices in these formulas are contragradient (i.e. inverse and transposed) to each other, explains the old-fashioned names for vectors and 1-forms (or covectors): *contra*variant and *co*variant vectors, respectively. Nevertheless, one should be careful: (A.1.7) represents the transformation of n components of one vector, whereas (A.1.6) encrypts the transformation of n-different 1-forms.

A.1.2 Tensors of type $\begin{bmatrix} p \\ q \end{bmatrix}$

A tensor is a multilinear map of a product of vector and dual vector spaces into the real numbers. An alternative definition of tensors specifies the transformation law of their components with respect to a change of the basis.

The related concepts of a vector and a 1-form can be generalized to objects of higher rank. The prototype of such an object is the stress *tensor* of continuum mechanics. A tensor T on V of type $\begin{bmatrix} p \\ q \end{bmatrix}$ is a multi-linear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{p} \times \underbrace{V \times \cdots \times V}_{q} \quad \to \quad \mathbb{R} . \tag{A.1.9}$$ deftensor

It can be described as a geometrical quantity whose components with respect to the cobasis ϑ^α and the basis e_β are given by

$$T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} = T\left(\vartheta^{\alpha_1}, \dots, \vartheta^{\alpha_p}; e_{\beta_1}, \dots, e_{\beta_q}\right) . \tag{A.1.10}$$ tensorcomps

The transformation law for tensor components can be deduced from (A.1.5) and (A.1.6):

$$T^{\alpha'_1 \dots \alpha'_p}{}_{\beta'_1 \dots \beta'_q} = L_{\alpha_1}{}^{\alpha'_1} \cdots L_{\alpha_p}{}^{\alpha'_p} \, L_{\beta'_1}{}^{\beta_1} \cdots L_{\beta'_q}{}^{\beta_q} \, T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} . \tag{A.1.11}$$ tentrafo

If we have two tensors, T of type $\begin{bmatrix} p \\ q \end{bmatrix}$ and S of type $\begin{bmatrix} r \\ s \end{bmatrix}$, we can construct its tensor product – the tensor $T \otimes S$ of type $\begin{bmatrix} p+r \\ q+s \end{bmatrix}$ defined as

$$\begin{aligned} &(T \otimes S)(\omega_1, \dots, \omega_{p+r}; v_1, \dots, v_{q+s}) \\ &= T(\omega_1, \dots, \omega_p; v_1, \dots, v_q) \, S(\omega_{p+1}, \dots, \omega_{p+r}; v_{q+1}, \dots, v_{q+s}) , \end{aligned} \tag{A.1.12}$$

for any one-forms ω and any vectors v.

Tensors of type $\begin{bmatrix} p \\ q \end{bmatrix}$ which have the form $v_1 \otimes \cdots \otimes v_p \otimes \omega_1 \otimes \cdots \otimes \omega_q$ are called *decomposable*. Each tensor is a linear combination of decomposable tensors. More precisely, using the definition of the components of T according to (A.1.11), one can prove that

$$T = T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}\, e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \otimes \vartheta^{\beta_1} \otimes \cdots \otimes \vartheta^{\beta_q} \,. \qquad \text{(A.1.13)}$$ decompose

Therefore tensor products of basis vectors e_α and of basis 1-forms ϑ^β constitute a basis of the vector space V^p_q of tensors of type $\left[{p \atop q}\right]$ on V. Thus the dimension of this vector space is n^{p+q}.

Elementary examples of the tensor spaces are given by the original vector space and its dual, $V^1_0 = V$ and $V^0_1 = V^*$. The Kronecker symbol δ^α_β is a tensor of type $\left[{1 \atop 1}\right]$, as can be verified with the help of (A.1.11).

A.1.3 $^\otimes$A generalization of tensors: geometric quantities

A geometric quantity is defined by the action of the general linear group on a certain set of elements. Important examples are tensor-valued forms, the orientation, and twisted tensors.

In field theory, tensors are not the only objects needed for the description of nature. Twisted forms or vector-valued forms, e.g., require a more general definition.

As we have seen above, there are two ways of dealing with tensors: either we can describe them as elements of the abstract tensor space V^p_q or as components, i.e. elements of $\mathbb{R}^{n^{p+q}}$, which have a prescribed transformation law. These observations can be generalized as follows:

Let W be a set, and let ρ be a left action of the group $GL(n,\mathbb{R})$ in the set W, i.e. to each element $\mathrm{L} \in GL(n,\mathbb{R})$ we attach a map $\rho_{\mathrm{L}} : W \to W$ in such a way that

$$\rho_{\mathrm{L}_1} \circ \rho_{\mathrm{L}_2} = \rho_{\mathrm{L}_1 \mathrm{L}_2} \qquad \mathrm{L}_1, \mathrm{L}_2 \in GL(n,\mathbb{R}). \qquad \text{(A.1.14)}$$ defrho

Denote by $P(V)$ the space of all bases of V and consider the Cartesian product $W \times P(V)$. The formula (A.1.5) provides us with a left action of $GL(n,\mathbb{R})$ in $P(V)$ which can be compactly written as $e' = \mathrm{L}e$; and then $\mathrm{L}_1(\mathrm{L}_2 e) = (\mathrm{L}_1\mathrm{L}_2)e$ holds. Thus we can define the left action of $GL(n,\mathbb{R})$ on the product $W \times P(V)$:

$$(w, e) \mapsto \big(\rho_{\mathrm{L}}(w), \mathrm{L}e\big)\,. \qquad \text{(A.1.15)}$$ laction

An orbit of this action is called an *geometric quantity* of type ρ on V. In other words, a geometric quantity of type ρ on V is an equivalence class $[(\omega, e)]$. Two pairs (w, e) and (w', e') are equivalent if and only if there exists a matrix $\mathrm{L} \in GL(n,\mathbb{R})$ such that

$$w' = \rho_{\mathrm{L}}(w) \qquad \text{and} \qquad e' = \mathrm{L}e\,. \qquad \text{(A.1.16)}$$ equiv

In many physical applications, the set W is an N-dimensional vector space $\mathbb{R}^N$ and it is required that the maps ρ_{L} are linear. In other words, ρ is a representation $\rho : GL(n,\mathbb{R}) \to GL(N,\mathbb{R})$ of the linear group $GL(n,\mathbb{R})$ in the vector space W by $N \times N$ matrices $\rho(L) = {\rho_A}^B(L) \in GL(N,\mathbb{R})$, with $A, B, \dots = 1, \dots, N$. Let us denote as e_A the basis of the vector space W. Then, we can represent the geometric quantity $w = w^A e_A$ by means of its components w^A with respect to the basis. The action of the group $GL(n,\mathbb{R})$ in W results in a linear transformation

$$e_A \longrightarrow e_{A'} \;=\; {\rho_{A'}}^B(L)\, e_B, \qquad \text{(A.1.17)}$$ GQbasis

and, accordingly, the components of the geometric quantity transform as

$$w^A \longrightarrow w^{A'} \;=\; {\rho_B}^{A'}(L^{-1})\, w^B. \qquad \text{(A.1.18)}$$ GQcomp

Examples:

1) We can take $W = V^p_q$ and $\rho_{\mathrm{L}} = \mathrm{id}_W$. The corresponding geometric quantity is then a tensor of type $\left[{p \atop q}\right]$.

2) We can take $W = \mathbb{R}^{n^{p+q}}$ and choose ρ in such a way that (A.1.16) will induce (A.1.11) together with (A.1.5). This type of geometric quantity is also a tensor of type $\left[{p \atop q}\right]$. For instance, if we take $W = \mathbb{R}^n$ and either $\rho_{\mathrm{L}} = \mathrm{L}$ or $\rho_{\mathrm{L}} = (\mathrm{L}^T)^{-1}$, then from (A.1.17) and (A.1.18) we get vectors (A.1.7) or 1-forms (A.1.8), respectively.

3) The two examples above can be combined. We can take $W = \mathbb{R}^{n^{p+q}} \otimes V^r_s$ and ρ as in example 2). That means that we can consider objects with components $T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q}$ belonging to V^r_s which transform according to the rule (A.1.11). This mixture of two approaches seems strange at first sight, but it appears productive if, instead of V^r_s, we take spaces of s-forms $\Lambda^s V$. Such *tensor-valued forms* turn out to be useful in differential geometry and physics.

4) Let $W = \{+1, -1\}$ and $\rho_{\mathrm{L}} = \mathrm{sgn}(\det \mathrm{L})$. This geometric quantity is an *orientation* in the vector space V. A frame $e \in P(V)$ is said to have a positive orientation, if it forms a pair with $+1 \in W$. Each vector space has two different orientations.

5) Combine the examples 1) and 4). Let $W = V^p_q$ and $\rho_{\mathrm{L}} = \mathrm{sgn}(\det \mathrm{L})\, \mathrm{id}_W$. This geometric quantity is called a *twisted* (or odd) tensor of type $\left[{p \atop q}\right]$ on V. Particularly useful are twisted exterior forms since they can be integrated even on a manifold which is non-orientable.

A.1.4 Almost complex structure

An *even-dimensional* vector space V, with $n = 2k$, can be equipped with an additional structure which finds many interesting applications in electrodynamics and in other physical theories. We say that a real vector space V has an *almost complex structure*[1] if a tensor $\mathbf{J}$ of type $\left[{1 \atop 1}\right]$ is defined on it which has the property

$$\mathbf{J}^2 = -1. \qquad \text{(A.1.19)}$$

almost1

With respect to a chosen frame, this tensor is represented by the components $\mathrm{J}_\alpha{}^\beta$ and the above condition is then rewritten as

$$\mathrm{J}_\alpha{}^\gamma\, \mathrm{J}_\gamma{}^\beta = -\delta^\beta_\alpha. \qquad \text{(A.1.20)}$$

almost2

[1]See Choquet et al. [4].

By means of the suitable choice of the basis e_α, the complex structure can be brought into the canonical form

$$\mathrm{J}_\alpha{}^\beta = \begin{pmatrix} 0 & -\mathbf{I}_k \\ \mathbf{I}_k & 0 \end{pmatrix} . \tag{A.1.21}$$ almost3

Here $\mathbf{I}_k$ is the k-dimensional unity matrix with $k = n/2$.

A.1.5 Exterior p-forms

Exterior forms are totally antisymmetric covariant tensors. Any tensor of type $\left[{0 \atop p}\right]$ defines an exterior p-form by means of the alternating map involving the generalized Kronecker.

As we saw at the beginning of Part A, exterior p-forms play a particular role as integrands in field theory. We will now turn to their general definition.

Let once more V be an n-dimensional linear vector space. An exterior p-form ω on V is a real-valued linear function

$$\omega : \underbrace{V \times V \times \ldots \times V}_{p \text{ factors}} \longrightarrow \mathbb{R} \tag{A.1.22}$$

such that

$$\omega(v_1, \ldots, v_\alpha, \ldots, v_\beta, \ldots, v_p) = -\,\omega(v_1, \ldots, v_\beta, \ldots, v_\alpha, \ldots, v_p) \tag{A.1.23}$$ exform

for all $v_1, \ldots, v_p \in V$ and for all $\alpha, \beta = 1, \ldots, p$. In other words, $\boldsymbol{\omega}$ is a completely antisymmetric tensor of type $\left[{0 \atop p}\right]$. In terms of a basis e_α of V and the cobasis ϑ^α of V^*, the linear function ω can be expressed as

$$\omega = \omega_{\alpha_1 \cdots \alpha_p}\, \vartheta^{\alpha_1} \otimes \ldots \otimes \vartheta^{\alpha_p} \,, \tag{A.1.24}$$ excomp

where each coefficient $\omega_{\alpha_1 \cdots \alpha_p} := \omega(e_{\alpha_1}, \ldots, e_{\alpha_p})$ is completely antisymmetric in all its indices.

The space of real-valued p-linear functions on V was denoted by V_p^0. Then, for any $\varphi \in V_p^0$, with

$$\varphi = \varphi_{\alpha_1 \cdots \alpha_p}\, \vartheta^{\alpha_1} \otimes \ldots \otimes \vartheta^{\alpha_p}\,, \tag{A.1.25}$$
lamb

we can define a corresponding (alternating) exterior p-form Alt φ by

$$\operatorname{Alt}\varphi = \varphi_{[\alpha_1 \cdots \alpha_p]}\, \vartheta^{\alpha_1} \otimes \ldots \otimes \vartheta^{\alpha_p}\,. \tag{A.1.26}$$
altlamb

Here we have

$$\varphi_{[\alpha_1 \cdots \alpha_p]} := \frac{1}{p!}\, \delta^{\beta_1 \ldots \beta_p}_{\alpha_1 \ldots \alpha_p}\, \varphi_{\beta_1 \cdots \beta_p} \tag{A.1.27}$$
anti

with the generalized Kronecker delta

$$\delta^{\beta_1 \ldots \beta_p}_{\alpha_1 \ldots \alpha_p} = \begin{cases} +1 & \text{if } \beta_1, \ldots, \beta_p \text{ is an even} \\ & \text{permutation of } \alpha_1, \ldots, \alpha_p\,, \\ -1 & \text{if } \beta_1, \ldots, \beta_p \text{ is an odd} \\ & \text{permutation of } \alpha_1, \ldots, \alpha_p\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{A.1.28}$$
delta

where $\alpha_1, \ldots, \alpha_p$ are p *different* numbers from the set $1, \ldots, n$. Provided $\varphi_{\alpha_1, \ldots, \alpha_p}$ is already antisymmetric in all its indices, then

$$\varphi_{[\alpha_1 \cdots \alpha_p]} = \varphi_{\alpha_1 \ldots \alpha_p}\,. \tag{A.1.29}$$
anti2

The set of exterior p-forms on V forms an $\binom{n}{p}$-dimensional subspace of V_p^0 which we denote by $\Lambda^p\, V^*$. Here $\binom{n}{p}$ represent the binomial coefficients. In particular, for $p = 0$ and $p = 1$, we shall have

$$\Lambda^0\, V^* = \mathbb{R}\,, \qquad \Lambda^1\, V^* = V^*\,. \tag{A.1.30}$$
lamb0

For $n = 4$, the dimensions of the spaces for p-forms are $\binom{4}{p} = \frac{4!}{p!(4-p)!}$, or

$$p = 0,\ 1,\ 2,\ 3,\ 4 \qquad \sim \qquad 1,\ 4,\ 6,\ 4,\ 1 \ \text{dimensions}\,, \tag{A.1.31}$$
dimforms

respectively, see Table A.1.5.

Table A.1.1: **Number of components** *of p-forms in 3 and 4 dimensions and examples from electrodynamics:* ρ electric charge and j electric current density, $\mathcal{D}$ electric and $\mathcal{H}$ magnetic excitation, E electric and B magnetic field, $\mathcal{A}$ covector potential, φ scalar potential, f gauge function; furthermore, L is the Lagrangian, $J = (\rho, j)$, $H = (\mathcal{D}, \mathcal{H})$, $F = (E, B)$, $A = (\varphi, \mathcal{A})$. Actually, the forms J and H are twisted forms, see Sec. A.2.6.

p-**form**	**n = 3**	**examples**	**n = 4**	**examples**
0-form	1	φ, f	1	f
1-form	3	$\mathcal{H}, E, \mathcal{A}$	4	A
2-form	3	$j, \mathcal{D}, B$	6	H, F
3-form	1	ρ	4	J
4-form	0	–	1	L
5-form	0	–	0	–

A.1.6 Exterior multiplication

The exterior product defines a $(p+q)$-form for every pair of p- and q-forms. The basis of the space of p-forms is then naturally constructed as the p-th exterior power of the 1-form basis. The exterior product converts the direct sum of all forms into an algebra.

In order to be able to handle exterior forms, we have to define their multiplication. The exterior product of the p 1-forms $\omega^1, \ldots, \omega^p \in V^*$ – taken in that order – is a p-form defined by

$$\omega^1 \wedge \ldots \wedge \omega^p := \delta^{1\ldots p}_{\alpha_1 \ldots \alpha_p}\, \omega^{\alpha_1} \otimes \ldots \otimes \omega^{\alpha_p}\,, \qquad \text{(A.1.32)}$$

exproduct

spoken as "omega-one wedge ... wedge omega-p". It follows that for any set of vectors $v_1, \dots, v_p \in V$,

$$(\omega^1 \wedge \dots \wedge \omega^p)(v_1, \dots, v_p) = \begin{vmatrix} \omega^1(v_1) & \omega^1(v_2) & \cdots & \omega^1(v_p) \\ \omega^2(v_1) & \omega^2(v_2) & \cdots & \omega^2(v_p) \\ \vdots & \vdots & \ddots & \vdots \\ \omega^p(v_1) & \omega^p(v_2) & \cdots & \omega^p(v_p) \end{vmatrix} . \tag{A.1.33}$$

exproduct2

Given $\omega \in \Lambda^p V^*$, so that

$$\omega = \omega_{\alpha_1 \dots \alpha_p} \vartheta^{\alpha_1} \otimes \dots \otimes \vartheta^{\alpha_p} , \qquad \text{with} \qquad \omega_{[\alpha_1 \dots \alpha_p]} = \omega_{\alpha_1 \dots \alpha_p} , \tag{A.1.34}$$

om1

we have

$$\begin{aligned} \omega &= \omega_{[\alpha_1 \dots \alpha_p]} \vartheta^{\alpha_1} \otimes \dots \otimes \vartheta^{\alpha_p} \\ &= \frac{1}{p!} \delta^{\beta_1 \dots \beta_p}_{\alpha_1 \dots \alpha_p} \omega_{\beta_1 \dots \beta_p} \vartheta^{\alpha_1} \otimes \dots \otimes \vartheta^{\alpha_p} , \end{aligned} \tag{A.1.35}$$

om2

and hence, cf. (20),

$$\boxed{\omega = \frac{1}{p!} \omega_{\beta_1 \dots \beta_p} \vartheta^{\beta_1} \wedge \dots \wedge \vartheta^{\beta_p} .} \tag{A.1.36}$$

om3

Since, in addition, the $\binom{n}{p}$ p-forms $\{\vartheta^{\alpha_1} \wedge \dots \wedge \vartheta^{\alpha_p},\ 1 \le \alpha_1 < \alpha_2 < \dots < \alpha_p \le n\}$ are linearly independent, it follows that they constitute a basis for $\Lambda^p V^*$. Equation (A.1.36) may also be written as

$$\omega = \sum_{\beta_1 < \beta_2 < \dots \beta_p} \omega_{\beta_1 \dots \beta_p} \vartheta^{\beta_1} \wedge \dots \wedge \vartheta^{\beta_p} . \tag{A.1.37}$$

om4

The indices $\beta_1 < \beta_2 < \dots < \beta_p$ are called *strongly ordered.* Furthermore, it is clear from (A.1.36) that a p-form, with $p > n$, is equal to zero.

The exterior product of two arbitrary forms is introduced as a map

$$\wedge : \Lambda^p V^* \times \Lambda^q V^* \longrightarrow \Lambda^{p+q} V^* \tag{A.1.38}$$

as follows: Let $\psi \in \Lambda^p V^*$ and $\phi \in \Lambda^q V^*$. Then $\psi \wedge \phi \in \Lambda^{p+q} V^*$ is defined by

$$\psi \wedge \phi = \frac{(p+q)!}{p!\,q!}\,\mathrm{Alt}(\psi \otimes \phi)\,. \tag{A.1.39}$$

arbex

In terms of a 1-form basis ϑ^α of V^*, we shall have

$$\begin{aligned} \psi &= \frac{1}{p!}\,\psi_{\beta_1\ldots\beta_p}\,\vartheta^{\beta_1}\wedge\ldots\wedge\vartheta^{\beta_p}\,,\\ \phi &= \frac{1}{q!}\,\phi_{\beta_1\ldots\beta_q}\,\vartheta^{\beta_1}\wedge\ldots\wedge\vartheta^{\beta_q}\,, \end{aligned} \tag{A.1.40}$$

arbex2

and their exterior product reads

$$\boxed{\psi \wedge \phi = \frac{1}{p!\,q!}\,\psi_{[\beta_1\ldots\beta_p}\,\phi_{\beta_{p+1}\ldots\beta_{p+q}]}\,\vartheta^{\beta_1}\wedge\ldots\wedge\vartheta^{\beta_{p+q}}\,.} \tag{A.1.41}$$

arbex3

From the definition (A.1.32), it is a straightforward matter to derive the following properties of exterior multiplication:

1) $(\lambda + \mu)\wedge\nu = \lambda\wedge\nu + \mu\wedge\nu$ [distributive law],

2) $(a\lambda)\wedge\nu = \lambda\wedge(a\nu) = a(\lambda\wedge\nu)$ [multiplicative law],

3) $(\lambda\wedge\nu)\wedge\omega = \lambda\wedge(\nu\wedge\omega)$ [associative law],

4) $\lambda\wedge\nu = (-1)^{pq}\,(\nu\wedge\lambda)$ [(anti)commutative law],

where $\lambda,\ \mu \in \Lambda^p V^*$, $\nu \in \Lambda^q V^*$, $\omega \in \Lambda^r V^*$, and $a \in \mathbb{R}$.

With the exterior multiplication introduced, the direct sum of the spaces of all forms

$$\Lambda^* V := \oplus_{p=0}^{n}\,\Lambda^p V \tag{A.1.42}$$

becomes an algebra over V. This is usually called the *exterior algebra*.

A.1.7 Interior multiplication of a vector with a form

> *The interior product decreases the rank of an exterior form by one.*

By exterior multiplication, we *in*crease the rank of a form. Besides this 'constructive' operation, we need a 'destructive' operation *de*creasing the rank of a form. Here interior multiplication comes in.

For $p > 0$, the interior product is a map

$$\lrcorner\ : V \times \Lambda^p V^* \longrightarrow \Lambda^{p-1} V^* \tag{A.1.43}$$

which is introduced as follows: Let $v \in V$ and $\phi \in \Lambda^p V^*$. Then $(v \lrcorner \boldsymbol{\phi}) \in \Lambda^{p-1} V^*$ is defined by

$$(v \lrcorner \phi)(u_1, \ldots, u_{p-1}) := \phi(v, u_1, \ldots, u_{p-1}) \,, \tag{A.1.44}$$ definner

for all $u_1, \ldots, u_{p-1} \in V$. We speak it as "$v$ *in* ϕ". In the literature sometimes the interior product of v and ϕ is alternatively abbreviated as $i_v \phi$. For $p = 0$,

$$v \lrcorner \phi := 0 \,. \tag{A.1.45}$$ in0

Note that, if $p = 1$, the definition (A.1.44) implies

$$v \lrcorner \phi = \phi(v) \,. \tag{A.1.46}$$ in1

The following properties of interior multiplication follow immediately from the definitions (A.1.44), (A.1.45):

1) $v \lrcorner (\phi + \psi) = v \lrcorner \phi + v \lrcorner \psi$ [distributive law],

2) $(v + u) \lrcorner \phi = v \lrcorner \phi + u \lrcorner \phi$ [linearity in a vector],

3) $(av) \lrcorner \phi = a(v \lrcorner \phi)$ [multiplicative law],

4) $v \lrcorner u \lrcorner \phi = -u \lrcorner v \lrcorner \phi$ [anticommutative law],

5) $v \lrcorner (\phi \wedge \omega) = (v \lrcorner \phi) \wedge \omega + (-1)^p \phi \wedge (v \lrcorner \omega)$ [(anti)Leibniz rule], where $\phi,\ \psi \in \Lambda^p V^*, \omega \in \Lambda^q V^*,\ v,\ u \in V$, and $a \in \mathbb{R}$.

Let e_α be a basis of V, ϑ^α the cobasis of V^*, then, by (A.1.46),

$$e_\alpha \lrcorner \vartheta^\beta = \vartheta^\beta(e_\alpha) = \delta^\beta_\alpha\,. \qquad \text{(A.1.47)}$$

indual

Hence, if we apply the vector basis e_β to the p-form

$$\psi = \frac{1}{p!}\,\psi_{\alpha_1\ldots\alpha_p}\,\vartheta^{\alpha_1}\wedge\ldots\wedge\vartheta^{\alpha_p}\,, \qquad \text{(A.1.48)}$$

psi

i.e. $e_\beta \lrcorner \psi$, then the properties listed above yield

$$\boxed{e_\beta \lrcorner \psi = \frac{1}{(p-1)!}\,\psi_{\beta\alpha_2\ldots\alpha_p}\,\vartheta^{\alpha_2}\wedge\ldots\wedge\vartheta^{\alpha_p}\,.} \qquad \text{(A.1.49)}$$

inbasis

If we multiply this formula by ϑ^β, we find the identity

$$\vartheta^\beta \wedge (e_\beta \lrcorner \psi) = p\psi\,. \qquad \text{(A.1.50)}$$

contr

A.1.8 $^{\otimes}$Volume elements on a vector space, densities, orientation

A volume element is a form of maximal rank. Thus, it has one nontrivial component. Under the action of the linear group, this component is a density of weight +1. Orientation is an equivalence class of volume forms related by a positive real factor. The choice of an orientation is equivalent to the selection of similarly oriented bases in V.

The space $\Lambda^n V^*$ of exterior n-forms on an n dimensional vector space V is 1-dimensional and, for $\boldsymbol{\omega} \in \Lambda^n V^*$, we have

$$\omega = \frac{1}{n!}\,\omega_{\alpha_1\ldots\alpha_n}\,\vartheta^{\alpha_1}\wedge\ldots\wedge\vartheta^{\alpha_n} = \omega_{1\ldots n}\,\vartheta^1\wedge\ldots\wedge\vartheta^n\,. \qquad \text{(A.1.51)}$$

exom

The nonzero elements of $\Lambda^n V^*$ are called *volume elements*.

Consider a linear transformation (A.1.5) of the basis e_α of V. The corresponding transformation of the cobasis ϑ^α of V^* is given by (A.1.6).

Let $L := \det\bigl(L_{\alpha'}{}^{\beta}\bigr)$. Then

$$\begin{aligned}\vartheta^{1'} \wedge \ldots \wedge \vartheta^{n'} &= L_{\alpha_1}{}^{1'} \ldots L_{\alpha_n}{}^{n'}\; \vartheta^{\alpha_1} \wedge \ldots \wedge \vartheta^{\alpha_n} \\ &= \left(L_{\alpha_1}{}^{1'} \ldots L_{\alpha_n}{}^{n'}\; \delta^{\alpha_1 \ldots \alpha_n}_{1 \ldots n}\right)\, \vartheta^{1} \wedge \ldots \wedge \vartheta^{n}\,.\end{aligned} \tag{A.1.52}$$

Hence, for the volume element[2], we have:

$$\boxed{\vartheta^{1'} \wedge \ldots \wedge \vartheta^{n'} = \det(L_{\beta}{}^{\alpha'})\; \vartheta^{1} \wedge \ldots \wedge \vartheta^{n}\,.} \tag{A.1.53}$$
volel

Since, in terms of the basis $\vartheta^{\alpha'}$,

$$\omega = \omega_{1' \ldots n'}\, \vartheta^{1'} \wedge \ldots \wedge \vartheta^{n'}\,, \tag{A.1.54}$$
exom2

it follows from (A.1.51) and (A.1.53) that

$$\omega_{1 \ldots n} = \det(L_{\beta}{}^{\alpha'})\, \omega_{1' \ldots n'} \; = L^{-1}\, \omega_{1' \ldots n'}\,, \tag{A.1.55}$$
exom3

with $L^{-1} = \det(L_{\beta}{}^{\alpha'})$, and, conversely,

$$\omega_{1' \ldots n'} = \det(L_{\alpha'}{}^{\beta})\, \omega_{1 \ldots n} = \; L\, \omega_{1 \ldots n}\,. \tag{A.1.56}$$
exom4

The geometric quantity with transformation law given by (A.1.56) is called a *scalar density.* It can easily be generalized. The geometric quantity $\mathcal{S}$ with transformation law

$$\mathcal{S}' = \det(L_{\alpha'}{}^{\beta})^{w}\, \mathcal{S} \tag{A.1.57}$$
deltrafo

is called scalar *density of weight* w. The generalization to tensor densities of weight w in terms of components, see (A.1.11), reads

$$\mathcal{T}^{\alpha' \cdots}{}_{\beta' \ldots} = \det(L_{\alpha'}{}^{\beta})^{w}\, L_{\alpha}{}^{\alpha'} \cdots L_{\beta'}{}^{\beta} \cdots \mathcal{T}^{\alpha \ldots}{}_{\beta \ldots}\,, \tag{A.1.58}$$
densityT

whereas *twisted* tensor densities of weight w on the right hand side pick up an extra factor $\operatorname{sgn}\det(L_{\alpha'}{}^{\beta})$.

Let ω and φ be two arbitrary volume forms on V. We will say that volumes are equivalent, if a positive real number $a > 0$ exists, such that $\varphi = a\,\omega$. This definition divides the space $\Lambda^{n} V^{*}$ into the two equivalence classes of n-forms. One calls each of the

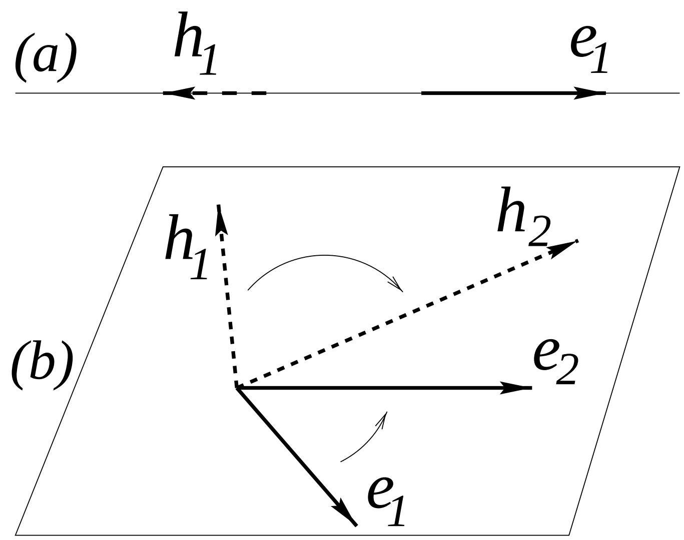

Figure A.1.4: Distinguishing orientation (a) on a line and (b) on a plane: The vector bases e_α and h_α are differently oriented.

equivalence classes an *orientation* on V, because a specification of a volume form uniquely determines a class of oriented bases on V and conversely. This can be demonstrated as follows:

On the intuitive level, in the case of a straight line one speaks of an orientation by distinguishing between positive and negative directions, whereas in a plane one chooses positive and negative values of an angle, see Fig. A.1.4. The generalization of these intuitive ideas to an n-dimensional vector space is contained in the concept of *orientation*: one says that two bases e_α and h_α of V are *similarly oriented* if $h_\alpha = A_\alpha{}^\beta\, e_\beta$, with $\det\left(A_\alpha{}^\beta\right) > 0$. This is clearly an equivalence relation which divides the space of all bases of V into two classes. An *orientation of the vector space* V is an equivalence class of ordered bases, and V is called *oriented vector space* when a choice of orientation is made.

[2]For the definition of a volume element *without* the use of a metric, see Synge and Schild [32], and, in particular, Laurent [14].

Now we return to the volume elements in V. Given the volume ω, we can define the function

$$o_\omega(e) := \operatorname{sgn} \omega(e_1, \ldots, e_n) \tag{A.1.59}$$ orient

on the set of all bases of V. It has only two values: $+1$ and -1, and accordingly we have a division of the set of all bases into the two subsets. One class is constituted of the bases for which $o_\omega(e) = 1$, and on the second class we have $o_\omega(e) = -1$. In each subset, the bases are similarly oriented.

In order to show this, let us assume that a volume form (A.1.51) is chosen and fixed, and let us take an arbitrary cobasis ϑ^α. The value of the volume form on the vectors of the basis e_α reads $\omega(e_1, \ldots, e_n) = \omega_{1\ldots n}(\vartheta^1 \wedge \cdots \wedge \vartheta^n)(e_1, \ldots, e_n) = \omega_{1\ldots n}$. Suppose this number is positive, then in accordance with (A.1.59) we have $o_\omega(e) = +1$. For a different basis $h_\alpha = A_\alpha{}^\beta\, e_\beta$ we obtain $\omega(h_1, \ldots, h_n) = \det\left(A_\alpha{}^\beta\right) \omega(e_1, \ldots, e_n) = \det\left(A_\alpha{}^\beta\right)\, \omega_{1\ldots n}$. Consequently, if the basis h_α is in the same subset as e_α, that is $o_\omega(h) = +1$, then $\det\left(A_\alpha{}^\beta\right) > 0$, which means that the bases e_α and h_α are similarly oriented. Conversely, assuming $\det\left(A_\alpha{}^\beta\right) > 0$ for any two bases $h_\alpha = A_\alpha{}^\beta\, e_\beta$, we find that (A.1.59) holds true for both bases.

Clearly, every volume form which is obtained by a "rescaling" $\omega_{1\ldots n} \to \varphi_{1\ldots n} = a\, \omega_{1\ldots n}$ with a positive factor a will define the same orientation function (A.1.59): $o_\omega(e) = o_{a\omega}(e)$. This yields the whole class of equivalent volume forms which we introduced at the beginning of our discussion.

The *standard orientation* of V for an arbitrary basis e_α is determined by the volume form $\vartheta^1 \wedge \cdots \wedge \vartheta^n$ with cobasis ϑ^α. A simple reordering of the vectors (for example, an interchange of the first and the second leg) of a basis may change the orientation.

A.1.9 ⊗Levi-Civita symbols and generalized Kronecker deltas

The Levi-Civita symbols are numerically invariant quantities and close relatives of the volume form. They can arise by applying the exterior product $\wedge$ *or the interior product* $\rfloor$ *n-times, respectively. Levi-Civita symbols are totally antisymmetric tensor densities, and their products can be expressed in terms of the generalized Kronecker delta.*

Volume forms provide a natural definition of very important tensor densities, the Levi-Civita symbols. In order to describe them, let us choose an arbitrary cobasis ϑ^α, and consider the form of maximal rank

$$\hat{\epsilon} := \vartheta^1 \wedge \ldots \wedge \vartheta^n. \qquad \text{(A.1.60)}$$ elemvol

We will call this *an elementary volume*. Recall that the transformation law of this form is given by (A.1.53), which means that $\hat{\epsilon}$ is the *n-form density of the weight* -1. By simple inspection it turns out that the wedge product

$$\vartheta^{\alpha_1} \wedge \ldots \wedge \vartheta^{\alpha_n} \qquad \text{(A.1.61)}$$ prodN

is either zero (when at least two of the wedge-factors are the same) or it is equal to $\hat{\epsilon}$ up to a sign. The latter holds when all the wedge-factors are different, and the sign is determined by the number of permutations which are needed for bringing the product (A.1.61) to the ordered form (A.1.60). This suggests a natural definition of the object $\epsilon^{\alpha_1\cdots\alpha_n}$ which has similar symmetry properties. That is, we define it by the relation

$$\vartheta^{\alpha_1} \wedge \ldots \wedge \vartheta^{\alpha_n} =: \epsilon^{\alpha_1\cdots\alpha_n}\,\hat{\epsilon}. \qquad \text{(A.1.62)}$$ epsilon1

As one can immediately check, the *Levi-Civita symbol* $\epsilon^{\alpha_1\cdots\alpha_n}$ can be expressed in terms of the generalized Kronecker symbol (A.1.28)[3]:

$$\epsilon^{\alpha_1\ldots\alpha_n} = \delta^{\alpha_1\ldots\alpha_n}_{1\,\ldots\,n}. \qquad \text{(A.1.63)}$$ eps-del1

[3] See Sokolnikoff [29].

In particular, we see that the only nontrivial component is $\epsilon^{1\dots n} = 1$. With respect to the change of the basis, this quantity transforms as the $\left[{n \atop 0}\right]$-valued 0-form density of the weight $+1$:

$$\epsilon^{\alpha'_1\dots\alpha'_n} = \det(L_{\alpha'}{}^{\beta})\, L_{\alpha_1}{}^{\alpha'_1}\dots L_{\alpha_n}{}^{\alpha'_n}\,\epsilon^{\alpha_1\dots\alpha_n}. \tag{A.1.64}$$

Recalling the definition of the determinant, we see that the components of the Levi-Civita symbol have *the same numerical values* with respect to all bases,

$$\epsilon^{\alpha'_1\dots\alpha'_n} = \epsilon^{\alpha_1\dots\alpha_n}. \tag{A.1.65}$$ epsinv

They are $+1, -1$, or 0.

Another fundamental antisymmetric object can be obtained from the elementary volume $\hat\epsilon$ with the help of the interior product operator. As we have learned from Sec. A.1.7, the interior product of a vector with a p-form generates a $(p-1)$-form. Thus, starting with the elementary volume n-form, and using the vector of the basis e_α, we find an $(n-1)$-form

$$\hat\epsilon_\alpha := e_\alpha \lrcorner\, \hat\epsilon. \tag{A.1.66}$$ epsA1

The transformation law of this object defines it as a *covector-valued $(n-1)$-form density of the weight* -1:

$$\hat\epsilon_{\alpha'} = \det(L_{\alpha'}{}^{\beta})^{-1}\, L_{\alpha'}{}^{\alpha}\,\hat\epsilon_\alpha. \tag{A.1.67}$$ transeps

Applying once more the interior product of the basis to (A.1.66), one obtains and $(n-2)$-form, etc.. Thus, we can construct the chain of forms:

$$\hat\epsilon_{\alpha_1\alpha_2} := e_{\alpha_2} \lrcorner\, \hat\epsilon_{\alpha_1}, \tag{A.1.68}$$

$$\vdots \tag{A.1.69}$$

$$\hat\epsilon_{\alpha_1\dots\alpha_n} := e_{\alpha_n} \lrcorner\, \dots\, \lrcorner\, e_{\alpha_1}\,\hat\epsilon. \tag{A.1.70}$$ epsilon2

The last object is a zero-form. The property 4) of the interior product forces all these epsilons to be totally antisymmetric in all their indices. Similarly to (A.1.67), we can verify that for $p = 0,\dots,n$ the object $\hat\epsilon_{\alpha_1\dots\alpha_p}$ is a $\left[{0 \atop p}\right]$-valued $(n-p)$-form density of the weight -1. These forms $\{\hat\epsilon, \hat\epsilon_{\alpha}, \hat\epsilon_{\alpha_1\alpha_2}, \ldots, \hat\epsilon_{\alpha_1\ldots\alpha_n}\}$, alternatively to $\{\vartheta^{\alpha}, \vartheta^{\alpha_1}\wedge\vartheta^{\alpha_2}, \ldots \vartheta^{\alpha_1}\wedge\ldots\vartheta^{\alpha_n}\}$, can be used as a basis for arbitrary forms in the exterior algebra $\Lambda^{*}V$.

In particular, we find that (A.1.70) is the $\left[{0 \atop n}\right]$-valued 0-form density of the weight -1. This quantity is also called the Levi-Civita symbol because of its evident similarity to (A.1.62). Analogously to (A.1.63) we can express (A.1.70) in terms of the generalized Kronecker symbol (A.1.28):

$$\hat\epsilon_{\alpha_1\ldots\alpha_n} = \delta^{\,1\ldots n}_{\alpha_1\ldots\alpha_n}. \tag{A.1.71}$$ eps-del2

Thus we find again that the only nontrivial component is $\hat\epsilon_{1\ldots n} = +1$. Note that despite the deep similarity, we cannot identify the two Levi-Civita symbols in the absence of the metric, hence the different notation (with and without hat) is appropriate.

It is worthwhile to derive a useful identity for the product of the two Levi-Civita symbols:

$$\begin{aligned}
\epsilon^{\alpha_1\ldots\alpha_n}\,\hat\epsilon_{\beta_1\ldots\beta_n} &= \hat\epsilon^{\alpha_1\ldots\alpha_n}\; e_{\beta_n}\rfloor\;\ldots\;\rfloor\, e_{\beta_1}\,\epsilon \\
&= e_{\beta_n}\rfloor\;\ldots\;\rfloor\, e_{\beta_1}\left(\vartheta^{\alpha_1}\wedge\ldots\wedge\vartheta^{\alpha_n}\right) \\
&= \left(\vartheta^{\alpha_1}\wedge\ldots\wedge\vartheta^{\alpha_n}\right)\left(e_{\beta_1},\ldots,e_{\beta_n}\right) \\
&= \begin{vmatrix} \delta^{\alpha_1}_{\beta_1} & \cdots & \delta^{\alpha_1}_{\beta_n} \\ \vdots & \ddots & \vdots \\ \delta^{\alpha_n}_{\beta_1} & \cdots & \delta^{\alpha_n}_{\beta_n} \end{vmatrix} = \delta^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n}.
\end{aligned} \tag{A.1.72}$$ epseps

The whole derivation is based just on the use of the corresponding definitions. Namely, we use (A.1.70) in the first line, (A.1.62) in the second line, (A.1.44) in the third one, and finally (A.1.33) in the last line. This identity helps a lot in calculations of the different contractions of the Levi-Civita symbols. For example, we easily obtain from (A.1.72):

$$\epsilon^{\alpha_1\ldots\alpha_p\gamma_1\ldots\gamma_{n-p}}\,\hat\epsilon_{\beta_1\ldots\beta_p\gamma_1\ldots\gamma_{n-p}} = (n-p)!\,\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p}. \tag{A.1.73}$$ epseps1

Furthermore, let us take an integer $q < p$. The contraction of (A.1.72) over the $(n-q)$ indices yields the same result (A.1.73) with p replaced by q. Comparing the two contractions, we then deduce for the generalized Kroneckers:

$$\delta^{\alpha_1\dots\alpha_q\gamma_{q+1}\dots\gamma_p}_{\beta_1\dots\beta_q\gamma_{q+1}\dots\gamma_p} = \frac{(n-q)!}{(n-p)!}\,\delta^{\alpha_1\dots\alpha_q}_{\beta_1\dots\beta_q}. \tag{A.1.74}$$

deldel

In particular, we find

$$\delta^{\alpha_1\dots\alpha_p}_{\alpha_1\dots\alpha_p} = \frac{n!}{(n-p)!}. \tag{A.1.75}$$

deldel1

Let us collect for a vector space of 4 dimensions the decisive formulas for going down the p-form ladder by starting from the 4-form density $\hat{\epsilon}$ and arriving at the 0-form $\hat{\epsilon}_{\alpha\beta\gamma\delta}$:

$$\begin{aligned}
\hat{\epsilon}_{\alpha} &= e_{\alpha} \rfloor \hat{\epsilon} = \hat{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^{\beta}\wedge\vartheta^{\gamma}\wedge\vartheta^{\delta}/3!\,,\\
\hat{\epsilon}_{\alpha\beta} &= e_{\beta} \rfloor \hat{\epsilon}_{\alpha} = \hat{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^{\gamma}\wedge\vartheta^{\delta}/2!\,,\\
\hat{\epsilon}_{\alpha\beta\gamma} &= e_{\gamma} \rfloor \hat{\epsilon}_{\alpha\beta} = \hat{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^{\delta}\,,\\
\hat{\epsilon}_{\alpha\beta\gamma\delta} &= e_{\delta} \rfloor \hat{\epsilon}_{\alpha\beta\gamma}\,.
\end{aligned} \tag{A.1.76}$$

epsilonhats

Going up the ladder yields:

$$\begin{aligned}
\vartheta^{\alpha}\wedge\hat{\epsilon}_{\beta\gamma\delta\mu} &= \delta^{\alpha}_{\mu}\,\hat{\epsilon}_{\beta\gamma\delta} - \delta^{\alpha}_{\delta}\,\hat{\epsilon}_{\beta\gamma\mu} + \delta^{\alpha}_{\gamma}\,\hat{\epsilon}_{\beta\delta\mu} - \delta^{\alpha}_{\beta}\,\hat{\epsilon}_{\gamma\delta\mu}\,,\\
\vartheta^{\alpha}\wedge\hat{\epsilon}_{\beta\gamma\delta} &= \delta^{\alpha}_{\delta}\,\hat{\epsilon}_{\beta\gamma} + \delta^{\alpha}_{\gamma}\,\hat{\epsilon}_{\delta\beta} + \delta^{\alpha}_{\beta}\,\hat{\epsilon}_{\gamma\delta}\,,\\
\vartheta^{\alpha}\wedge\hat{\epsilon}_{\beta\gamma} &= \delta^{\alpha}_{\gamma}\,\hat{\epsilon}_{\beta} - \delta^{\alpha}_{\beta}\,\hat{\epsilon}_{\gamma}\,,\\
\vartheta^{\alpha}\wedge\hat{\epsilon}_{\beta} &= \delta^{\alpha}_{\beta}\,\hat{\epsilon}\,.
\end{aligned} \tag{A.1.77}$$

One can, with respect to the $\hat{\epsilon}$-system, define a (pre-metric) duality operator $^{\diamond}$ which establishes an equivalence between p-forms and totally antisymmetric tensor densities of weight $+1$ and of type $\left[{n-p \atop 0}\right]$. In terms of the bases of the corresponding linear spaces, this operator is introduced as

$$^{\diamond}(\hat{\epsilon}_{\alpha_1\dots\alpha_p}) := \delta^{\beta_1\dots\beta_p}_{\alpha_1\dots\alpha_p}\,e_{\beta_1}\otimes\cdots\otimes e_{\beta_p}. \tag{A.1.78}$$

diamondef

Consequently, given an arbitrary p-form ω expanded with respect to the $\hat{\epsilon}$-basis as

$$\omega = \frac{1}{(n-p)!}\,\omega^{\alpha_1\dots\alpha_{n-p}}\,\hat{\epsilon}_{\alpha_1\dots\alpha_{n-p}}, \tag{A.1.79}$$

diamond1

the map $^\diamond$ defines a tensor density by

$$^\diamond\omega := \frac{1}{(n-p)!}\,\omega^{\alpha_1\dots\alpha_{n-p}}\,e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{n-p}}. \qquad \text{(A.1.80)}$$

diamond2

For example, in $n=4$ we have $^\diamond\hat\epsilon_\alpha = e_\alpha$ and $^\diamond\hat\epsilon = 1$. Thus, every 3-form $\varphi = \varphi^\alpha\,\hat\epsilon_\alpha$ is mapped into a vector density $^\diamond\varphi = \varphi^\alpha\,e_\alpha$, whereas a 4-form ω yields a scalar density $^\diamond\omega$.

A.1.10 The space M^6 of two-forms in four dimensions

Electromagnetic excitation and field strength are both 2-forms. On the 6-dimensional space of 2-forms, there exists a natural 6-metric, which is an important property of this space.

Let e_α be an arbitrary basis of V, with $\alpha,\beta,\dots = 0,1,2,3$. In later applications, the zeroth leg e_0 can be related to the time coordinate of spacetime, but this will not always be the case (for the null symmetric basis (C.2.14), e.g., all the e_α's have the same status with respect to time). The three remaining legs will be denoted by e_a, with $a,b,\dots = 1,2,3$. Accordingly, the dual basis of V^* is represented by $\vartheta^\alpha = (\vartheta^0, \vartheta^a)$.

In the linear space of 2-forms $\Lambda^2 V^*$, every element can be decomposed according to $\varphi = \frac{1}{2}\,\varphi_{\alpha\beta}\,\vartheta^\alpha\wedge\vartheta^\beta$. The basis $\vartheta^\alpha\wedge\vartheta^\beta$ consists of 6 simple 2-forms. This 6-plet can be alternatively numbered by a collective index. Accordingly, we enumerate the antisymmetric index pairs $01, 02, 03, 23, 31, 12$ by uppercase letters $I, J...$ from 1 to 6:

$$
\begin{aligned}
\text{Б}^I = \begin{pmatrix} \text{Б}^1 \\ \text{Б}^2 \\ \text{Б}^3 \\ \text{Б}^4 \\ \text{Б}^5 \\ \text{Б}^6 \end{pmatrix} &= \begin{pmatrix} \vartheta^0 \wedge \vartheta^1 \\ \vartheta^0 \wedge \vartheta^2 \\ \vartheta^0 \wedge \vartheta^3 \\ \vartheta^2 \wedge \vartheta^3 \\ \vartheta^3 \wedge \vartheta^1 \\ \vartheta^1 \wedge \vartheta^2 \end{pmatrix} \\
&= \begin{pmatrix} \vartheta^0 \wedge \vartheta^a \\ \frac{1}{2} \hat{\epsilon}_{bcd} \, \vartheta^c \wedge \vartheta^d \end{pmatrix} = \begin{pmatrix} \beta^a \\ \hat{\epsilon}_b \end{pmatrix} \qquad \text{(A.1.81)}
\end{aligned}
$$

beh1

With the Б^I as basis (speak 'cyrillic B' or 'Beh'), we can set up a 6-dimensional vector space $M^6 := \Lambda^2 V^*$. This vector space will play an important role in our considerations in Parts D and E. The extra decomposition with respect to β^a and $\hat{\epsilon}_b$ will be convenient for recognizing where the electric and where the magnetic pieces of the field are located.

We denote the elementary volume 3-form by $\hat{\epsilon} = \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$. Then $\hat{\epsilon}_a = e_a \rfloor \hat{\epsilon}$ is the basis 2-form in the space spanned by the 3-coframe ϑ^a, see (A.1.66). This notation has been used in (A.1.81). Moreover, as usual, the 1-form basis then can be described by $\hat{\epsilon}_{ab} = e_b \rfloor \hat{\epsilon}_a$. Some useful algebraic relations can be immediately derived:

$$\hat{\epsilon}_a \wedge \vartheta^b = \delta_a^b \, \hat{\epsilon}, \qquad \text{(A.1.82)}$$

$$\hat{\epsilon}_{ab} \wedge \vartheta^c = \delta_a^c \, \hat{\epsilon}_b - \delta_b^c \, \hat{\epsilon}_a, \qquad \text{(A.1.83)}$$

$$\hat{\epsilon}_{ab} \wedge \hat{\epsilon}_c = \hat{\epsilon} \, \hat{\epsilon}_{abc}. \qquad \text{(A.1.84)}$$

Correspondingly, taking into account that $\vartheta^0 \wedge \hat{\epsilon} =: \text{Vol}$ is the elementary 4-volume in V, we find

$$\beta^a \wedge \beta^b = 0, \qquad \text{(A.1.85)}$$ bb0

$$\hat{\epsilon}_a \wedge \hat{\epsilon}_b = 0, \qquad \text{(A.1.86)}$$ ee0

$$\hat{\epsilon}_a \wedge \beta^b = \delta_a^b \, \text{Vol}, \qquad \text{(A.1.87)}$$ epsbe

$$\hat{\epsilon}_{ab} \wedge \beta^c \wedge \vartheta^d = \left(- \delta_a^c \, \delta_b^d + \delta_b^c \, \delta_a^d \right) \text{Vol}. \qquad \text{(A.1.88)}$$

Every 2-form, being an element of M^6, can now be represented as $\varphi = \varphi_I\, \mathcal{B}^I$ by its 6 components with respect to the basis (A.1.81). A 4-form ω, i.e., a form of the maximal rank in four dimensions, is expanded with respect to the wedge products of the $\mathcal{B}$-basis as $\omega = \frac{1}{2}\omega_{IJ}\, \mathcal{B}^I \wedge \mathcal{B}^J$. The coefficients ω_{IJ} form a *symmetric* 6×6 matrix since the wedge product between 2-forms is evidently commutative.

A 4-form has only one component. This simple observation enables us to introduce a natural metric on the 6-dimensional space M^6 as the symmetric bilinear form

$$\boldsymbol{\varepsilon}(\omega,\, \varphi) := (\omega \wedge \varphi)(e_0, e_1, e_2, e_3), \qquad \omega, \varphi \in M^6, \qquad \text{(A.1.89)}$$ 6met

where e_α is a vector basis. Although the metric (A.1.89) apparently depends on the choice of the basis, the linear transformation $e_{\alpha'} \to L_{\alpha'}{}^{\alpha}\, e_\alpha$ induces the pure rescaling $\boldsymbol{\varepsilon} \to \det(L_{\alpha'}{}^{\alpha})\, \boldsymbol{\varepsilon}$. Using the expansion of the 2-forms with respect to the bivector basis $\mathcal{B}^I$, the bilinear form (A.1.89) turns out to be

$$\boldsymbol{\varepsilon}(\omega,\, \varphi) = \omega_I\, \varphi_J\, \varepsilon^{IJ}, \quad \text{where} \quad \varepsilon^{IJ} = (\mathcal{B}^I \wedge \mathcal{B}^J)(e_0, e_1, e_2, e_3). \qquad \text{(A.1.90)}$$ met6co

A direct inspection by using the definition (A.1.81) and the identity (A.1.87) shows that the 6-metric components read explicitly

$$\varepsilon^{IJ} = \begin{pmatrix} 0 & \mathbf{I}_3 \\ \mathbf{I}_3 & 0 \end{pmatrix}. \qquad \text{(A.1.91)}$$ Levi1

Here $\mathbf{I}_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the 3×3 unit matrix. Thus we see that the metric (A.1.89) is always non-degenerate. Its signature is $(+,+,+,-,-,-)$. Indeed, the eigenvalues λ of the matrix (A.1.91) are defined by the characteristic equation $\det(\varepsilon^{IJ} - \lambda\, \delta^{IJ}) = (\lambda^2 - 1)^3 = 0$. The symmetry group which preserves the 6-metric (A.1.89) is isomorphic to $O(3,3)$.

By construction, the elements of (A.1.91) numerically coincide with the components of the Levi-Civita symbol ϵ^{ijkl}, see

(A.1.63):

$$\varepsilon^{IJ} = \epsilon_{IJ} := \begin{pmatrix} 0 & \mathbf{I}_3 \\ \mathbf{I}_3 & 0 \end{pmatrix} . \qquad \text{(A.1.92)}$$ Levi1a

Similarly, the covariant Levi-Civita symbol $\hat{\epsilon}_{mnpq}$, see (A.1.71), can be represented in 6D notation by the matrix

$$\hat{\epsilon}_{IK} = \hat{\epsilon}_{KI} := \begin{pmatrix} 0 & \mathbf{I}_3 \\ \mathbf{I}_3 & 0 \end{pmatrix} . \qquad \text{(A.1.93)}$$ Levi2

One can immediately prove by multiplying the matrices (A.1.91) and (A.1.93) that their product is equal the 6D-unity, in complete agreement with (A.1.73). Thus, the Levi-Civita symbols can be consistently used for raising and lowering indices in M^6.

Transformation of the M^6–basis

What happens in M^6 when the basis in V is changed, $e_\alpha \to e_{\alpha'}$? As we know, such a change is described by the linear transformation (A.1.5). Then the cobasis transforms in accordance with (A.1.6):

$$\vartheta^\alpha = L_{\alpha'}{}^\alpha \, \vartheta^{\alpha'} \, . \qquad \text{(A.1.94)}$$ dualtrafo1

In the $(1+3)$-matrix form, this can be written as

$$\begin{pmatrix} \vartheta^0 \\ \vartheta^a \end{pmatrix} = \begin{pmatrix} L_0{}^0 & L_b{}^0 \\ L_0{}^a & L_b{}^a \end{pmatrix} \begin{pmatrix} \vartheta'^0 \\ \vartheta'^b \end{pmatrix} . \qquad \text{(A.1.95)}$$ dualtrafo2

Correspondingly, the 2-form basis (A.1.81) transforms into a new bivector basis

$$\mathcal{B}'^I = \begin{pmatrix} \beta'^a \\ \hat{\epsilon}'_b \end{pmatrix} , \qquad \text{(A.1.96)}$$ beh2

$$\beta'^a = \begin{pmatrix} \vartheta'^0 \wedge \vartheta'^1 \\ \vartheta'^0 \wedge \vartheta'^2 \\ \vartheta'^0 \wedge \vartheta'^3 \end{pmatrix} , \qquad \hat{\epsilon}'_b = \begin{pmatrix} \vartheta'^2 \wedge \vartheta'^3 \\ \vartheta'^3 \wedge \vartheta'^1 \\ \vartheta'^1 \wedge \vartheta'^2 \end{pmatrix} .$$

Substituting (A.1.94) into (A.1.81), we find that the new and old 2-form bases are related by induced linear transformation

$$\begin{pmatrix} \beta^a \\ \hat{\epsilon}_a \end{pmatrix} = \begin{pmatrix} P^a{}_b & W^{ab} \\ Z_{ab} & Q_a{}^b \end{pmatrix} \begin{pmatrix} \beta'^b \\ \hat{\epsilon}'_b \end{pmatrix}, \qquad \text{(A.1.97)} \quad \texttt{B2B}$$

where

$$P^a{}_b = L_0{}^0 L_b{}^a - L_0{}^a L_b{}^0, \quad Q_b{}^a = (\det L_c{}^d)\,(L^{-1})_b{}^a, \qquad \text{(A.1.98)} \quad \texttt{PQ}$$

$$W^{ab} = L_c{}^0 L_d{}^a \,\epsilon^{bcd}, \qquad Z_{ab} = \hat{\epsilon}_{acd} L_0{}^c L_b{}^d. \qquad \text{(A.1.99)} \quad \texttt{MN}$$

The 3×3 matrix $(L^{-1})_b{}^a$ is inverse to the 3×3 sub-block $L_a{}^b$ in (A.1.94). The inverse transformation is easily computed:

$$\begin{pmatrix} \beta'^a \\ \hat{\epsilon}'_a \end{pmatrix} = \frac{1}{\det L} \begin{pmatrix} Q_b{}^a & W^{ba} \\ Z_{ba} & P^b{}_a \end{pmatrix} \begin{pmatrix} \beta^b \\ \hat{\epsilon}_b \end{pmatrix}, \qquad \text{(A.1.100)} \quad \texttt{invB2B}$$

where the determinant of the transformation matrix (A.1.94) reads

$$\det L := \det L_{\alpha'}{}^{\beta} = \left[L_0{}^0 - L_a{}^0 (L^{-1})_b{}^a L_0{}^b\right] \det L_c{}^d. \qquad \text{(A.1.101)} \quad \texttt{detLam}$$

One can write an arbitrary linear transformation $L \in GL(4,\mathbb{R})$ as a product

$$L = L_1\, L_2\, L_3 \qquad \text{(A.1.102)} \quad \texttt{LLL}$$

of three matrices of the form

$$L_1 = \begin{pmatrix} 1 & U_b \\ 0 & \delta^a_b \end{pmatrix}, \qquad \text{(A.1.103)} \quad \texttt{sub1}$$

$$L_2 = \begin{pmatrix} 1 & 0 \\ V^a & \delta^a_b \end{pmatrix}, \qquad \text{(A.1.104)} \quad \texttt{sub2}$$

$$L_3 = \begin{pmatrix} \Lambda_0{}^0 & 0 \\ 0 & \Lambda_b{}^a \end{pmatrix}. \qquad \text{(A.1.105)} \quad \texttt{sub3}$$

Here $V^a, U_b, \Lambda_0{}^0, \Lambda_b{}^a$, with $a,b = 1,2,3$, describe $3{+}3{+}1{+}9 = 16$ elements of an arbitrary linear transformation. The matrices $\{L_3\}$ form the group $\mathbb{R}\otimes GL(3,\mathbb{R})$ which is a subgroup of

$GL(4,\mathbb{R})$, whereas the sets of unimodular matrices $\{L_1\}$ and $\{L_2\}$ evidently form two *Abelian* subgroups in $GL(4,\mathbb{R})$.

In the study of the covariance properties of various objects in M^6, it is thus sufficient to consider the three separate cases (A.1.103)-(A.1.105). Using (A.1.98) and (A.1.99), we find for $L = L_1$

$$P^a{}_b = Q_b{}^a = \delta^a_b, \qquad W^{ab} = \epsilon^{abc}\,U_c, \qquad Z_{ab} = 0. \quad \text{(A.1.106)}$$ case1

Similarly, for $L = L_2$ we have

$$P^a{}_b = Q_b{}^a = \delta^a_b, \qquad W^{ab} = 0, \qquad Z_{ab} = \hat{\epsilon}_{abc}\,V^c, \quad \text{(A.1.107)}$$ case2

and for $L = L_3$

$$P^a{}_b = \Lambda_0{}^0\,\Lambda_b{}^a, \qquad Q_b{}^a = (\det\Lambda)(\Lambda^{-1})_b{}^a, \qquad W^{ab} = Z_{ab} = 0. \quad \text{(A.1.108)}$$ case3

A.1.11 Almost complex structure on M^6

An almost complex structure on the space of 2-forms determines a splitting of the complexification of M^6 into two invariant 3-dimensional subspaces.

Let us introduce an almost complex structure $\mathbf{J}$ on the M^6. We recall that every tensor of type $\left[{1 \atop 1}\right]$ represents a linear operator on a vector space. Accordingly, if $\varphi \in M^6$, it is of type $\left[{0 \atop 1}\right]$ and $\mathbf{J}(\varphi)$ can be defined as a contraction. The result will be again an element of the M^6. By definition, $\mathbf{J}(\mathbf{J}(\varphi)) = -\mathbf{I}_6\,\varphi$ or

$$\mathbf{J}^2 = -1, \quad \text{(A.1.109)}$$ dualdef2

see (A.1.20).

As a tensor of type $\left[{1 \atop 1}\right]$, the operator $\mathbf{J}$ can be represented as a 6×6 matrix. Since the basis in M^6 is naturally split into $3+3$ parts in (A.1.81), we can write it in terms of the set of four 3×3 matrices,

$$\mathrm{J}_I{}^K = \begin{pmatrix} C^a{}_b & A^{ab} \\ B_{ab} & D_a{}^b \end{pmatrix}. \quad \text{(A.1.110)}$$ almostIJ

Because of (A.1.109), the 3×3 blocks A, B, C, D are constrained by

$$\begin{aligned} &A^{ac}B_{cb} + C^a{}_c C^c{}_b = -\delta^a_b, \\ &C^a{}_c A^{cb} + A^{ac} D_c{}^b = 0, \\ &B_{ac} C^c{}_b + D_a{}^c B_{cb} = 0, \\ &B_{ac} A^{cb} + D_a{}^c D_c{}^b = -\delta^b_a. \end{aligned} \qquad \text{(A.1.111)}$$ almostclose

Complexification of the M^6

An almost complex structure on the M^6 motivates a *complex* generalization of the M^6 to the *complexified* linear space $M^6(\mathbb{C})$. The elements of the $M^6(\mathbb{C})$ are the complex 2-forms $\omega \in M^6(\mathbb{C})$, i.e. their components ω_I in a decomposition $\omega = \omega_I B^I$ are complex. Alternatively, one can consider $M^6(\mathbb{C})$ as a real 12-dimensional linear space spanned by the basis (B^I, iB^I), where i is the imaginary unit. We will denote by $\overline{M^6}(\mathbb{C})$ the complex conjugate space.

The same symmetric bilinear form as in (A.1.89) defines also a natural metric in $M^6(\mathbb{C})$. Note however, that now an orthogonal (complex) basis can be always introduced in $M^6(\mathbb{C})$ so that $\varepsilon^{IJ} = \delta^{IJ}$ in that basis.

Incidentally, one can define another scalar product on a complex space $M^6(\mathbb{C})$ by

$$\boldsymbol{\varepsilon}'(\omega, \varphi) := (\omega \wedge \overline{\varphi})(e_0, e_1, e_2, e_3), \qquad \omega, \varphi \in M^6. \quad \text{(A.1.112)}$$ 6metric1

The significant difference between these two metrics is that $\boldsymbol{\varepsilon}'$ assigns a real length to any complex vector, whereas $\boldsymbol{\varepsilon}$ defines complex vector lengths.

We will assume that the $\mathbf{J}$ operator is defined in $M^6(\mathbb{C})$ by the same formula $\mathbf{J}(\omega)$ as in M^6. In other words, $\mathbf{J}$ remains a *real* linear operator in $M^6(\mathbb{C})$, i.e. for every complex 2-form $\omega \in M^6(\mathbb{C})$ one has $\overline{\mathbf{J}(\omega)} = \mathbf{J}(\overline{\omega})$. The eigenvalue problem for the operator $\mathbf{J}(\omega_\lambda) = \lambda\, \omega_\lambda$ is meaningful only in the complexified space $M^6(\mathbb{C})$ because, in view of the property (A.1.109), the eigenvalues are $\lambda = \pm i$. Each of these two eigenvalues has multiplicity 3, which follows from the reality of $\mathbf{J}$. Note that the 6×6

matrix of the $\mathbf{J}$ operator has 6 eigenvectors, but the number of eigenvectors with eigenvalue $+i$ is equal to the number of eigenvectors with eigenvalue $-i$ because they are complex conjugate to each other. Indeed, let $\mathbf{J}(\omega) = i\omega$, then the conjugation yields $\overline{\mathbf{J}(\omega)} = \mathbf{J}(\overline{\omega}) = -i\overline{\omega}$.

Let us denote the 3-dimensional subspaces of $M^6(\mathbb{C})$ which correspond to the eigenvalues $+i$ and $-i$ by

$$\overset{(s)}{M} := \left\{\omega \in M^6(\mathbb{C}) \mid \mathbf{J}(\omega) = i\omega\right\}, \tag{A.1.113}$$

subspace

$$\overset{(a)}{M} := \left\{\varphi \in M^6(\mathbb{C}) \mid \mathbf{J}(\varphi) = -i\varphi\right\}, \tag{A.1.114}$$

respectively. Evidently, $\overline{\overset{(s)}{M}} = \overset{(a)}{M}$. Therefore, we can restrict our attention only to the self-dual subspace. This will be assumed in our derivations from now on.

Accordingly, every form ω can be decomposed into a self-dual and an anti-self-dual piece[4],

$$\omega = \overset{(s)}{\omega} + \overset{(a)}{\omega}, \tag{A.1.115}$$

selfantiself2

with

$$\begin{aligned} \overset{(s)}{\omega} &= \frac{1}{2}\left[\omega - i\mathbf{J}(\omega)\right], \\ \overset{(a)}{\omega} &= \frac{1}{2}\left[\omega + i\mathbf{J}(\omega)\right]. \end{aligned} \tag{A.1.116}$$

selfantiself1

It can be checked that $\mathbf{J}(\overset{(s)}{\omega}) = +i\ \overset{(s)}{\omega}$ and $\mathbf{J}(\overset{(a)}{\omega}) = -i\ \overset{(a)}{\omega}$.

[4] A discussion of the use of self dual and anti-self dual 2-forms in general relativity can be found in Kopczyński and Trautman [13], e.g..

A.1.12 Computer algebra

Also in electrodynamics, research usually requires the application of computers. Besides numerical methods and visualization techniques, the manipulation of formulas by means of "computer algebra" systems is nearly a must. By no means are these methods confined to pure algebra, also differentiations and integrations, for example, can be executed with the help of computer algebra tools.

"If we do work on the *foundations* of classical electrodynamics, we can dispense with computer algebra," some true fundamentalists will claim. Is this really true? Well, later, as soon as we will analyze the Fresnel equation in Sec. D.1.4, we couldn't have done it to the extent we really did without using an efficient computer algebra system. Thus, our fundamentalist is well advised if she or he is going to learn some computer algebra. Accordingly, along with our introducing of some mathematical tools in exterior calculus, we will mention computer algebra systems like *Reduce*[5], *Maple*[6], and *Mathematica*[7] – and we will specifically explain of how to apply the Reduce package *Excalc*[8] to the exterior forms immanent in electrodynamics.

In practical work in solving problems by means of computer algebra, it is our experience that it is best to have access to different computer algebra systems. Even though in the course of time good features of one system "migrated" to other systems, still, for a certain specified purpose one system may be better suited than another one — and for different purposes these may

[5]Hearn [8] created this Lisp-based system. For introductions into Reduce, see Toussaint [36], Grozin [6], MacCallum and Wright [15], or Winkelmann and Hehl [38]; in the latter text you can learn of how to get hold of a Reduce system for your computer. Reduce as applied to general-relativistic field theories is described, e.g., by McCrea [16] and by Socorro et al. [28]. In our presentation, we partly follow the lectures of Toussaint [36].

[6]Maple, written in C, was created by a group at the University of Waterloo, Canada. A good introduction is given by Char et al. [3].

[7]Wolfram, see [39], created the C-based Mathematica software package which is in very wide-spread use.

[8]Schrüfer [25, 26] is the creator of that package, cf. also [27]. Excalc is applied to Maxwell's theory by Puntigam et al. [22].

Figure A.1.5: "Here is the new Reduce-update on a hard disk."

be different systems. There does not exist as yet *the* optimal system for all purposes. Therefore, it is not only on rare occasions that we have to feed the results of a calculation by means of one system as input into another system.

For computations in electrodynamics, relativity, and gravitation, we keep the three general-purpose computer algebra systems Reduce, Maple, and Mathematica. Other systems are available[9]. Our workhorse for corresponding calculations in exterior calculus is the Reduce-package Excalc, but also in the *MathTensor* package[10] of Mathematica exterior calculus is implemented. For the manipulation of tensors we use the following packages: In

[9] In the review of Hartley [7] possible alternative systems are discussed, see also Heinicke and Hehl [11].

[10] Parker and Christensen [21] created this package; for a simple application see Tsantilis [37].

Reduce the library of McCrea[11] and *GRG*[12], in Maple *GRTensorII*[13], and in Mathematica, besides MathTensor, the *Cartan* package[14].

Computer algebra systems are almost exclusively interactive systems nowadays. If one is installed in your computer, you can call the system usually by typing in its name or an abbreviation therefrom, i.e., '`reduce`', '`maple`', or '`math`', and then hitting the return key, or you have to click the corresponding icon. In the case of 'reduce', the system introduces itself and issues a '`1:`'. It waits for your first command. A command is a statement, usually some sort of expression, a part of a formula or a formula, followed by a terminator[15]. The latter is a semicolon `;` if you want to see the answer of the system, otherwise a dollar sign `$`. Reduce is case insensitive, i.e., the lower case letter `a` is not distinguished from the upper case letter `A`.

Formulating Reduce input

As an input statement to Reduce, we type in a certain legitimately formed expression. This means that, with the help of some operators, we compose formulas according to well-defined rules. Most of the built-in operators of Reduce, like the arithmetic operators `+` (plus), `-` (minus), `*` (times), `/` (divided by), `**` (to the power of)[16] are self-explanatory. They are so-called *infix operators* since the are positioned *in* between their arguments. By means of them we can construct combined expressions of the type $(x+y)^2$ or $x^3 \sin x$, which in Reduce read `(x+y)**2` and `x**3*sin x`, respectively.

If the command

[11] See McCrea's lectures [16].

[12] The GRG system, created by Zhytnikov [40], and the $\mathrm{GRG_{EC}}$ system of Tertychniy [35, 34, 20] grew from the same root; for an application of $\mathrm{GRG_{EC}}$ to the Einstein-Maxwell equations, see [33].

[13] See the documentation of Musgrave et al. [19]. Maple applications to the Einstein-Maxwell system are covered in the lectures of McLenaghan [17].

[14] Soleng [30] is the creator of 'Cartan'.

[15] Has nothing to do with Arnold Schwarzenegger!

[16] Usually one takes the circumflex for exponentiation. However, in the Excalc package this operator is redefined and used as the wedge symbol for exterior multiplication.

```
(x+y)**2;
```

is executed, you will get the expanded form $x^2+2xy+y^2$. There is a so-called switch `exp` in Reduce that is usually switched on. You can switch it off by the command

```
off exp;
```

Type in again

```
(x+y)**2;
```

Now you will find that Reduce doesn't do anything and gives the expression back as it received it. With `on exp;` you can go back to the original status.

Using the switches is a typical way to influence Reduce's way of how to evaluate an expression. A partial list of switches is collected in a table on the next page.

Let us give some more examples of expressions with infix operators:

```
(u+v)*(y-x)/8
(a>b) and (c<d)
```

Here we have the logical and relational operators `and`, `>` (greater than), `<` (less than). Widely used are also the infix operators

```
neq   >=   <=   or   not   :=
```

`neq` means not equal. The assignment operator `:=` assigns the value of the expression on its right-hand-side (its second argument) to the identifier on its left-hand-side (its first argument). In Reduce, logical (or Boolean) expressions have only the truth values `t` (true) or `nil` (false). They are only allowed within certain statements (namely in `if`, `while`, `repeat`, and `let` statements) and in so-called rule lists.

A *prefix operator* stands in front of its argument(s). The arguments are enclosed by parentheses and separated by commas:

```
cos(x)
int(cos(x),x)
factorial(8)
```

	Switch	description if switch is on	example
*	`allfac`	factorize simple factors	$2x+2 \rightarrow 2(x+1)$
	`div`	divide by the denominator	$(x^2+2)/x \rightarrow x+2/x$
*	`exp`	expand all expressions	$(x+1)(x-1) \rightarrow x^2-1$
*	`mcd`	make (common) denominator	$x+x^{-1} \rightarrow (x^2+1)/x$
*	`lcm`	cancel least common multiples	
	`gcd`	cancel greatest common divisor	
	`rat`	display as polynomial in `factor`	$\frac{x+1}{x} \rightarrow 1+x^{-1}$
*	`ratpri`	display rationals as fraction	$1/x \rightarrow \frac{1}{x}$
*	`pri`	dominates `allfac, div, rat, revpri`	
	`revpri`	display polynomials in opposite order	$x^2+x+1 \rightarrow 1+x+x^2$
	`rounded`	calculate with floats	$1/3 \rightarrow 0.333333333333$
	`complex`	simplify complex expressions	$1/i \rightarrow -i$
	`nero`	don't display zero results	$0 \rightarrow$
*	`nat`	display in Reduce input format	$\frac{x^2}{3} \rightarrow x{*}{*}2/3$
	`msg`	suppress messages	
	`fort`	display in Fortran format	
	`tex`	display in TeX format	

Table A.1.2: Switches for Reduce's reformulation rules. Those marked with * are turned on by default, the other ones are off.

In ordinary notation, the second statement reads $\int \cos x\, dx$.

The following mathematical functions are built-in as prefix operators:

`sin`	`sind`	`cos`	`cosd`	`tan`	`tand`
`cot`	`cotd`	`sec`	`secd`	`csc`	`cscd`
`asin`	`asind`	`acos`	`acosd`	`atan`	`atand`
`acot`	`acotd`	`asec`	`asecd`	`acsc`	`acscd`
`sinh`	`cosh`	`tanh`	`coth`	`sech`	`csch`
`asinh`	`acosh`	`atanh`	`acoth`	`asech`	`acsch`
`sqrt`	`exp`	`ln`	`log`	`log10`	`logb`
`dilog`	`erf`	`expint`	`cbrt`	`abs`	`hypot`
`factorial`					

Identifiers ending with `d` indicate that this operator expects its argument expressed in degree. `log` and `ln` stand for the natural logarithm, but `ln` only has the numerical properties of `log`. `logb` is the logarithm to base n; accordingly n must be specified as a second argument of `logb`. `hypot` calculates the hypotenuse according to $\text{hypot}(x, y) = \sqrt{x^2 + y^2}$. `csc` is the cosecans, `dilog` the Euler dilogarithm with $\text{dilog}(z) = -\int_0^z \log(1-\zeta)/\zeta \, d\zeta$, `erf` the Gaussian error function with $\text{erf}(x) = 2/\sqrt{\pi} \int_0^x e^{-t^2} dt$, `abs` the absolute value function, `expint` the exponential integral with $\text{expint}(x) = \int_{-\infty}^x e^t/t \, dt$, and, eventually, `cbrt` the operator for the cubic root.

Reduce only knows a few elementary rules for these operators. In addition to these built-in rules and operators, the Reduce user may want to define her or his own rules and operators (i.e., functions) by calling the command `operator`. No arguments are specified in the declaration statement. After the declaration, the specified operators may be used with arguments, just like `sin`, `cos`, etc.:

```
clear f,k,m,n$
operator f$
f(m);
f(n):=n**4+h**3+p**2;
f(4,k):=g;
```

If an operator is given with a certain argument, say `f(n)`, and an expression (here `n**4+h**3+p**2+u`, which contains the argument `n` of the operator) is assigned to the operator `f(n)`, this is a specific assignment only. There is *no* general functional relationship established between the argument of the operator and the same identifier which appears in the assigned expression. Such a relationship can only be created by self-defined rules. Let us demonstrate this somewhat difficult point as follows:

```
f(n):=n**4+h**3+p**2$
f(k);                   % does not evaluate to the value
                        % k**4+h**3+p**2+u, but only to f(k)
f(n);                   % again yields n**4+h**3+p**2+u
```

A newly created operator, which has no previously assigned value, carries as value its own name with the arguments (in contrast to the elements of an array which are initialized with value zero and which can never have as values the array name with their indices!). These operators have no properties, unless `let` rules are specified.

All operators may

- have values assigned to, as in

```
log(u):=12$
cos(2*k*pi):=1$
```

- have properties declared for some collections of arguments (for example, the value of `sin(`*integer*`*pi)` is always 0),
- be fully defined, either by the user, or by Reduce, as is the case for the operator `df` for differentiation.

With operators defined so far, we are able to construct Reduce expressions combining variables and operators in such a way that they represent our mathematical formulas. Reduce distinguishes between three kinds of expressions: Integer, scalar, and Boolean.

Integer expressions evaluate to whole numbers, for example

```
2
9-6
5**7+9*(6-j)*(k+h)
```

provided the variables `j,k,h` evaluate to integers.

Scalar expressions consist of (syntactically correct) sequences of numbers, variables, operators, left and right parentheses, and commas and are the usual representation of mathematical expressions in Reduce:

```
sin(8*y**4)+h(u)-(a+b)**7
df(u,x,8)*pi
b(y)+factorial(9)
a
```

The minimal scalar expressions which are known to Reduce are variables or numbers. The following rules are applied on evaluation of scalar expressions :

- Variables and operators with a number of arguments have the algebraic value they were last assigned or, if never assigned, stand for themselves.
- Nevertheless, some special expressions, such as elements of arrays (indexed variables), initially have the value 0.
- Operators act according to the rules that are defined for them. Only if there is no matching rule, the operator with its argument stands for itself (`cos 0`, for example, evaluates to `1`, but `cos x` won't get evaluated, as long as `x` is an unbound variable). Note that an (inappropriate) assignment such as `cos(0):=7` will have the same effect as a rule that defines `cos(0)` to be 7.
- Procedures of expressions are evaluated with the values of their actual parameters used in the procedure call.
- The algebraic evaluation of expressions (also called simplification) is controlled by the switches which may be turned on or off by the Reduce user.
- In any case, the standard rules of algebra apply. Parentheses are allowed. Expressions may be combined with legal operators to build new expressions. Those new expressions take on the new value built from the values of the subexpressions via the operators and taking into account the control switches.

Examples:

```
clear a,b$                  % a and b are declared to be unbound
a*b;
pol;                        % still not assigned
pol:=(a+b)**3$              % now assigned
pol;
on gcd$                     % greatest common divisor switch on
off exp$                    % expansion switch off
```

```
pol;
f:=g*m*m/r**2;
on div$                    % removes identical factors in
                           % numerator and denominator
f;
off gcd,div$ on exp$       % reset switches
```

We didn't give the output. You should try to get this yourself on your computer.

Boolean expressions use the well–known Boolean algebra and have truth values `t` for true and `nil` for false. For handling of Boolean expressions we have already mentioned the Boolean infix operators. Boolean expressions are only allowed within `if`-, `while`-, or `repeat`-statements. Examples of typical Boolean expressions are

```
j neq 2
a=b and (d or g)
(a+7) > 18                   % if a evaluates to an integer
```

If you want to display the truth value of a Boolean expression, use the `if`-statement, as in the following example:

```
if  2**28 <  10**7
  then write "less"
  else write "greater or equal";
```

Rudiments of evaluation

A Reduce program is a follow-up of commands. And the evaluation of the commands may be conditioned by switches that we switch on or off (also by a command). Let us look into the evaluation process a bit closer. After a command has been sent to the computer by hitting the return key, the whole command is evaluated. Each expression is evaluated from left to right, and the values obtained are combined with the operators specified. Sub–statements or sub–expressions existing within other expressions, like in

```
clear g,x$
```

```
a:=sin(g:=(x+7)**6);
cos(n:=2)*df(x**10,x,n);
```

are always evaluated first. In the first case, the value of `(x+7)**6` is assigned to `g`, and then `sin((x+7)**6)` is assigned to `a`. Note that the value of a whole assignment statement is always the value of its right–hand–side. In the second case, Reduce assigns `2` to `n`, then computes `df(x**10,x,2)`, and eventually returns `90*x**8*cos(2)` as the value of the whole statement. Note that both of these examples represents bad programming style, which should be avoided.

One exception to the process of evaluation exists for the assignment operator `:=` . Usually, the arguments of an operator are evaluated before the operator is applied to its arguments. In an assignment statement, the left side of the assignment operator is *not* evaluated. Hence

```
clear b,c$
a:=b$
a:=c$
a;
```

will not assign `c` to `b`, but rather `c` to `a`.
The process of evaluation in an assignment statement can be studied in the following examples:

```
clear h$
g:=1$
a:=(g+h)**3$
a;              % yields: (1+h)**3
g:=7$
a;              % yields: (1+h)**3
```

After the second statement, the variable `a` hasn't the value `(g+h)**3`, but rather `(1+h)**3`. This doesn't change by the fifth statement either where a new value is assigned to `g`. As one will recognize, `a` still has the value of `(1+h)**3`. If we want `a` to depend on `g`, then we must assign `(g+h)**3` to `a` as long as `g` is still unbound:

```
clear g,h$
a:=(g+h)**3$     % all variables are still unbound
g:=1$
g:=7$
a;               % yields: (7+h)**3
```

Now a has the value of (7+h)**3 rather than (g+h)**3.

Sometimes it is necessary to remove the assigned value from a variable or an expression. This can be achieved by using the operator clear as in

```
clear g,h$
a:=(g+h)**3$
g:=1$
a;
clear g$
a;
```

or by overwriting the old value by means of a new assignment statement:

```
clear b,u,v$
a:=(u+v)**2$
a:=a-v**2$
a;
b:=b+1$
b;
```

The evaluation of a; results in the value u*(u+2*v), since (u+v)**2 had been assigned to a, and a-v**2 (i.e., (u+v)**2-v**2) was reassigned to a. The assignment b:=b+1; will, however, lead to a difficulty: Since no value was previously assigned to b, the assignment replaces b *literally* with b+1 (whereas the previous a:=a-v**2 statement produces the *evaluation* a:=(u+v)**2-v**2). The last evaluation b; will lead to an error or will even hang up the system, because b+1 is assigned to b. As soon as b is evaluated, Reduce returns b+1, whereby b still has the value b+1, and so on. Therefore the evaluation process leads to an infinite loop. Hence we should avoid such recursions.

Incidentally, if you want to finish a Reduce session, just type in `bye;`

After these glimpses on Reduce, we will turn to the real object of our interest.

Loading Excalc

We load the Excalc package by

```
load_package excalc$
```

The system will tell us that the operator $\wedge$ is redefined, since it became the new wedge operator.

Excalc is designed such that the input to the computer is the same as what would have been written down for a hand-calculation. For example, the statement `f*x∧y + u _|(y∧z∧x)` would be a legitimately built Excalc expression with $\wedge$ denoting the exterior and `_|` (underline followed by a vertical bar) the interior product sign. Note that *before* the interior product sign `_|` (spoken *in*) there must be a blank; the other blanks are optional.

However, before Excalc can understand our intentions, we better declare u to be a (tangential) vector

```
tvector u;
```

f to be a scalar (i.e. a zero-form), and x, y, z to be 1-forms:

```
pform f=0, x=1, y=1, z=1;
```

A variable that is not declared to be a vector or a form is treated as a constant; thus zero-forms must also be declared. After our declarations, we can input our command

```
f*x^y+u _|(y^z^x);
```

Of course, the system cannot do much with this expression, but it expands the interior product. It also knows, of course, that

```
u _|f;
```

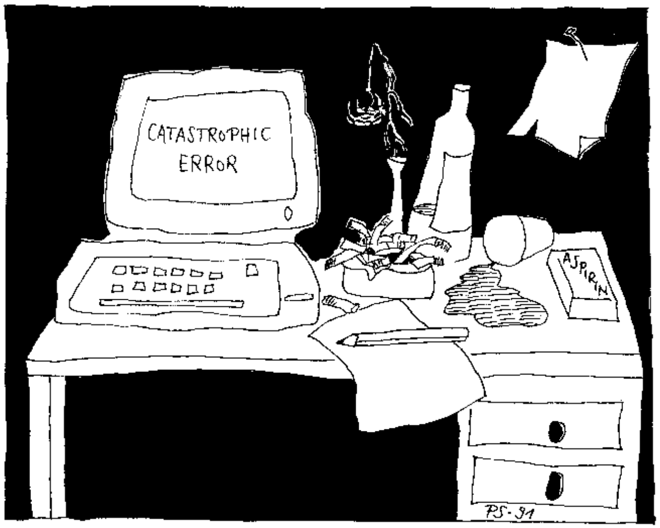

Figure A.1.6: “Catastrophic error,” a Reduce error message.

vanishes, that $y \wedge x = -x \wedge y$, or that $x \wedge x = 0$. If we want to check the rank of an expression, we can use

```
exdegree (x^y);
```

This yields 2 for our example.

Quite generally, Excalc can handle scalar-valued exterior forms, vectors and operations between them, as well as non-scalar valued forms (indexed forms). Simple examples of indexed forms are the Kronecker delta δ_α^β or the connection 1-form $\Gamma_\alpha{}^\beta$ of Sec. C.1.2. Their declaration reads

```
pform delta(a,b)=0, gamma1(a,b)=1;
```

The names of the indices are arbitrary. Subsequently, in the program a lower index is marked by a minus sign, an upper index with a plus (or with nothing), i.e., δ_1^1 translates into `delta(-1,1)` etc.

Excalc is a good tool for studying differential equations, for calculations in field theory and general relativity or for such simple things as calculating the Laplacian of a tensor field for an arbitrarily given frame. Excalc is completely embedded in

Reduce. Thus, all features and facilities of Reduce are available in a calculation.

If we declare the dimension of the underlying space by

```
spacedim 4;
```

then

```
pform a=2,b=3;   a^b;
```

yields `0`.

These are the fundamental commands of Excalc for exterior algebra. As soon as we will have introduced exterior calculus with frames and coframes, with vector fields and fields of forms – not to forget exterior and Lie differentiation, we will come back to Excalc and we will better appreciate its real power.

A.2

Exterior calculus

Having developed the concepts involved in the exterior *algebra* associated with an n-dimensional linear vector space V, we now look at how this structure can be 'lifted' onto an n-dimensional *differentiable manifold* X_n or, for short, onto X. The procedure for doing this is the same as for the transition from tensor algebra to tensor calculus. At each point x of X there is an n-dimensional vector space X_x, the tangent vector space at x. We identify the space X_x with the vector space V considered in the previous chapter. Then, at each point x, the exterior algebra of forms is determined on $V = X_x$. However, in differential geometry, one is concerned not so much with objects defined at isolated points as with *fields* over the manifold X or over open sets $U \subset X$. A field ω of p-forms on X is defined by assigning a p-form to each point x of X and, if this assignment is performed in a smooth manner, we shall call the resulting field of p-forms an *exterior differential p-form*. For simplicity we shall take 'smooth' to mean C^∞, although in physical applications the degree of differentiability may be less.

A.2.1 $^\otimes$Differentiable manifolds

A topological space becomes a differentiable manifold when an atlas of coordinate charts is introduced in it. Coordinate transformations are smooth in the intersections of the charts. The atlas is oriented when in all intersections the Jacobians of the coordinate transformations are positive.

In order to describe more rigorously how fields are introduced on X, we have to recall some basic facts about manifolds. At the start, one needs a *topological structure*. To be specific, we will normally assume that X is a connected, Hausdorff, and paracompact topological space. *Topology* on X is introduced by the collection of *open sets* $\mathcal{T} = \{U_\alpha \subset X | \alpha \in I\}$ which, by definition, satisfy the three conditions: (i) both, the empty set $\oslash$ and the manifold itself X belong to that collection, $\oslash, X \in \mathcal{T}$, (ii) any *union* of open sets is again open, i.e., $\bigcup_{\alpha\in J} U_\alpha \in \mathcal{T}$ for any subset $J \in I$, (iii) any *intersection of a finite* number of open sets is open, i.e., $\bigcap_{\alpha\in K} U_\alpha \in \mathcal{T}$ for any finite subset $K \in I$.

A topological space X is *connected*, if one cannot represent it by the sum $X = X_1 \bigcup X_2$, with open $X_{1,2}$ and $X_1 \bigcap X_2 = \oslash$. Usually, for a spacetime manifold, one further requires a *linear connectedness* which means that any two points of X can be connected by a continuous path. A topological space X is *Hausdorff* when for any two points $p_1 \neq p_2 \in X$ one can find open sets $p_1 \in U_1 \subset X$, $p_2 \in U_2 \subset X$ such that $U_1 \bigcap U_2 = \oslash$. Hausdorff's axiom forbids the 'branched' manifolds of the sort depicted in Fig. A.2.1. Finally, a connected Hausdorff manifold is *paracompact* when X can be covered by a countable number of open sets, i.e. $X = \bigcup_{\alpha\in K} U_\alpha$ for a countable subset $K \in I$.

A *differentiable manifold* is a topological space X plus a *differentiable structure* on it. The latter is defined as follows: A *coordinate chart* on X is a pair (U, ϕ), where $U \in \mathcal{T}$ is an open set and the map $\phi : U \to \mathbb{R}^n$ is a homeomorphism (i.e., continuous with a continuous inverse map) of U onto an open subset of the arithmetic space of n-tuples $\mathbb{R}^n$. This map assigns n labels or coordinates $\phi(p) = \{x^1(p), \dots, x^n(p)\}$ to any point $p \in U \subset X$.

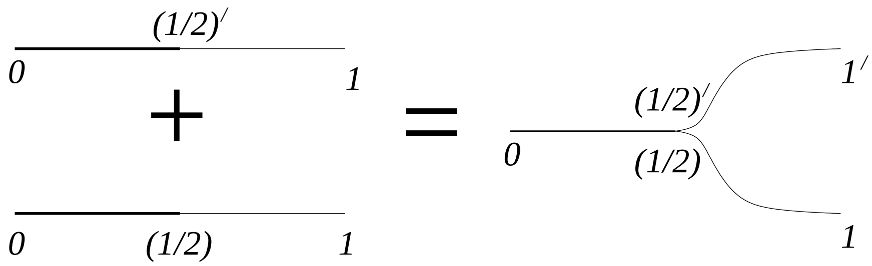

Figure A.2.1: *Non-Hausdorff* manifold: take two copies of the line segment $\{0,1\}$, and identify (paste together) their left halves *excluding* the points $(1/2)$ and $(1/2)'$. In the resulting manifold, the Hausdorff axiom is violated for the pair of points $(1/2)$ and $(1/2)'$.

Given any two intersecting charts, (U_α, ϕ_α) and (U_β, ϕ_β) with $U_\alpha \bigcap U_\beta \neq \oslash$, the map

$$f_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad \text{(A.2.1)}$$ cotra

is C^∞. The latter gives coordinate transformation in the intersection of charts. The whole collection of charts $\{(U_\alpha, \phi_\alpha)|\alpha \in I\}$ is called an *atlas* for every open covering of $X = \bigcup_{\alpha\in I} U_\alpha$. The two atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\alpha, \psi_\alpha)\}$ are said to be compatible if their union is again an atlas. Finally, the *differentiable structure* on X is a maximal atlas $\mathcal{A}(X)$ in the sense that its union with any atlas gives again $\mathcal{A}(X)$.

The atlas $\{(U_\alpha, \phi_\alpha)\}$ on X is said *oriented*, if all the transition functions (A.2.1) are orientation preserving, i.e., the corresponding Jacobian determinants are everywhere positive,

$$J(f_{\alpha\beta}) = \det\left(\frac{\partial x^i}{\partial y^j}\right) > 0, \qquad \text{(A.2.2)}$$ J

where $\phi_\alpha = \{x^i\}$ and $\phi_\beta = \{y^i\}$, with $i, j = 1, \ldots, n$. Then $f_{\alpha\beta} = (x^1(y^1, \ldots, y^n), \ldots, x^n(y^1, \ldots, y^n))$.

The differentiable manifold X is *orientable* if it has an oriented atlas. The notions of orientability and orientation on a manifold will prove to be very important in the theory of integration

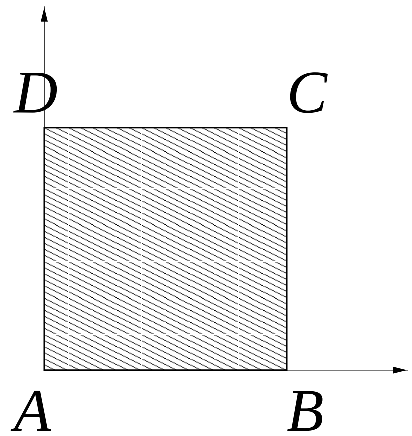

Figure A.2.2: Rectangle $ABCD$ in $\mathbb{R}^2$.

of differential forms. It is straightforward to provide examples of orientable and non-orientable manifolds. The following *two-dimensional* manifolds can be easily constructed with the help of the cut and paste techniques:

Consider an $\mathbb{R}^2$ and cut out a rectangle $ABCD$, as shown in Fig. A.2.2. As such, this is a compact two-dimensional manifold with boundary which is topologically equivalent to a disc.

However, after gluing together the sides of this rectangle, one can construct a number of compact manifolds *without a boundary*. The first example is obtained when we identify the opposite sides without twisting them, as shown in Fig. A.2.3. The resulting manifold is a two-dimensional torus $\mathbb{T}^2$ that is topologically equivalent to a sphere with one handle. This two-dimensional compact manifold is orientable.

Another possibility is to glue the rectangle $ABCD$ together after *twisting both pairs of opposite sides*, beforehand. This is shown in Fig. A.2.4. As a result, one obtains a *real projective plane* $\mathbb{P}^2$ which is represented by a sphere with a disc removed and the resulting hole is closed up by a "cross-cap", i.e. by identifying its diametrically opposite points (a more spectacular way is to say that a hole is closed by a Möbius strip). This two-dimensional manifold is also compact, but it is *non-orientable*.

Finally, we can glue the rectangle $ABCD$ with *twisting one pair of opposite sides* while matching the two other sides untwisted, as shown in Fig. A.2.5. The resulting compact two-dimensional manifold is a famous *Klein bottle* $\mathbb{K}^2$. The Klein bottle cannot be drawn in the $\mathbb{R}^3$ without self-intersections. However, it is possible to understand it as a sphere with *two* discs removed and the holes closed up with two "cross-caps" (the Möbius strips). The Klein bottle is also a compact but *non-orientable* manifold.

In general, one can prove that any connected compact two-dimensional manifold can be realized as a sphere with a certain number of small discs removed and a finite number of either handles or "cross-caps" attached in order to close the holes.

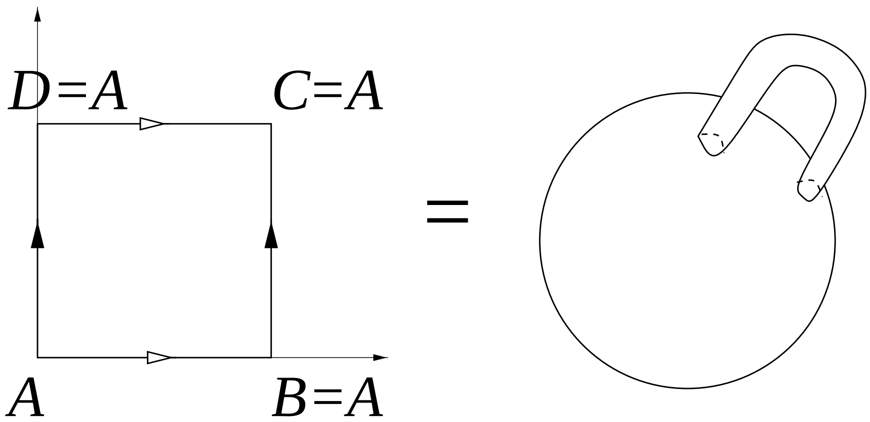

Figure A.2.3: Torus $\mathbb{T}^2$.

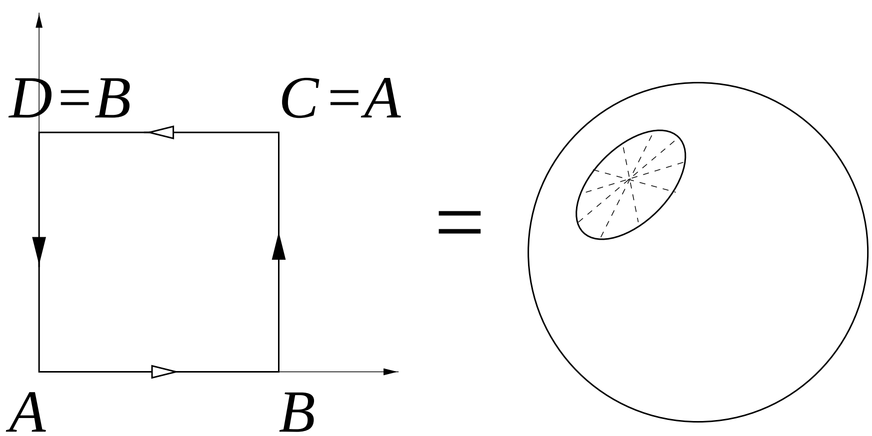

Figure A.2.4: Real projective plane $\mathbb{P}^2$.

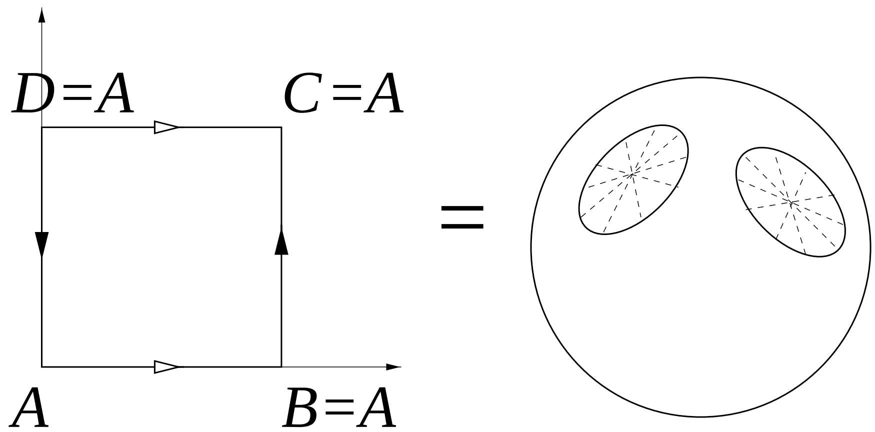

Figure A.2.5: Klein bottle $\mathbb{K}^2$.

A.2.2 Vector fields

Vector fields smoothly assign to each point of a manifold an element of the tangent space. The commutator of two vector fields is a new vector field.

Let us denote by $C(X)$ the algebra of differentiable functions on X. A *tangent vector* u at a point $x \in X$ is defined as an operator which maps $C(X)$ into $\mathbb{R}$ and satisfies the condition

$$u(fg) = f(x)\, u(g) + g(x)\, u(f), \qquad \forall f, g \in C(X). \tag{A.2.3}$$ vec1

A physical motivation comes from the notion of velocity. Indeed, let us consider a smooth curve $x(t)$ such that $0 \leq t \leq 1$ and $x(0) = x$. Then the directional derivative of a function $f \in C(X)$ along $x(t)$ at x,

$$v(f) = \left.\frac{df(x(t))}{dt}\right|_{t\,=\,0}, \tag{A.2.4}$$

is a linear mapping $v : C(X) \to \mathbb{R}$ satisfying (A.2.3). Choose a local coordinate system $\{x^i\}$ $(i = 1, \ldots, n)$ on a coordinate neighborhood $U \ni x$. Then the differential operator $\partial_i := \partial/\partial x^i$, for each i, satisfies (A.2.3). It can be demonstrated that the set of vectors $\{\partial_i\}$, $i = 1, \ldots, n$ provides a basis of the tangent space X_x at $x \in U$. We will use Latin letters to label coordinate indices.

A mapping u which assigns a tangent vector $u_x \in X_x$ to each point x is called a *vector field* on the manifold X. If we consider a smooth function $f(x)$ on X, then $u(f) := u_x(f)$ is a function on X. A vector field is called differentiable when a function $u(f)$ is differentiable for any $f \in C(X)$. In local coordinates $\{x^i\}$, a vector field is described $u = u^i(x)\,\partial_i$ by its components $u^i(x)$ which are smooth functions of coordinates.

For every two vector fields u and v a *commutator* $[u, v]$ is naturally defined by

$$[u, v](f) := u(v(f)) - v(u(f)). \qquad \text{(A.2.5)}$$ comm-uv

This is again a vector field. Please check that the condition (A.2.3) is satisfied. In local coordinates, because of $u = u^i(x)\,\partial_i$ and $v = v^i(x)\,\partial_i$, we find the components of the commutator $[u, v] = [u, v]^i(x)\,\partial_i$ as

$$[u, v]^i = u^j\, v^i{}_{,j} - v^j\, u^i{}_{,j}. \qquad \text{(A.2.6)}$$ comm-uv-i

A.2.3 One-form fields, differential p-forms

One-form fields assign to each point of a manifold an element of the dual tangent space. Differential p-forms are then defined pointwise as the exterior products of 1-form fields.

The dual vector space X_x^* is called *cotangent space* at x. The elements of X_x^* are 1-forms ω which map X_x into $\mathbb{R}$. An *exterior differential 1-form* ω is defined on X if a 1-form $\omega_x \in X_x^*$ is assigned to each point x. A natural example is provided by the differential *df* of a function f which is defined by

$$(df_x(u)) := u_x(f), \qquad \forall u \in X_x, \qquad \text{(A.2.7)}$$ dfun1

or, in local coordinates $\{x^i\}$,

$$df = f_{,i}\, dx^i\,. \qquad \text{(A.2.8)}$$ dfun2

Obviously, the coordinate differentials dx^i describe fields of 1-forms which provide a basis of X_x^* at each point. The bases $\{\partial_i\}$ and $\{dx^i\}$ are dual to each other, i.e., $dx^i(\partial_j) = \delta^i_j$. Repeating pointwise the constructions of the previous Sec. A.2.2, we find that a differential p-form ω on $U \subset X$ is expressible in the form

$$\omega = \frac{1}{p!}\,\omega_{i_1 \dots i_p}(x)\; dx^{i_1} \wedge \dots \wedge dx^{i_p}\,, \qquad (A.2.9)$$ expform

where $\omega_{i_1 \dots i_p}(x) = \omega_{[i_1 \dots i_p]}(x)$ are differentiable functions of the coordinates. The space of differential p-forms on X will be denoted by $\Lambda^p(X)$. In this context we may write $\Lambda^0(X) = C(X)$ for the set of differentiable functions on X.

A.2.4 Images of vectors and one-forms

The image of a vector field (an arrow) arises from the velocity of a point moving along an arbitrary curve. The prototype of an image of a 1-form (a pair of ordered hyperplanes) emerges from differential of a function.

It seems worthwhile to provide simple pictures for differential 1-form fields and vector fields. The physical prototype of a tangent vector v is a *velocity* of a particle moving along a given curve. Therefore a vector can pictorially be represented by an *arrow*, see Fig. A.2.6.

The prototype of a 1-form ω can be represented by the differential df of a function f: its components represent the gradient. Therefore, a suitable picture for a 1-form is given by *two parallel hyperplanes*, see Fig. A.2.6, which describe surfaces of constant value of f. With an arrowhead it is indicated in which direction the value of f is increasing. The "stronger" the 1-form is, the closer those two planes are. In physics, a generic 1-form is the *wave* 1-form defined as the gradient of the phase (think of a de Broglie wave!), or the *momentum* 1-form, defined as the gradient of the action function.

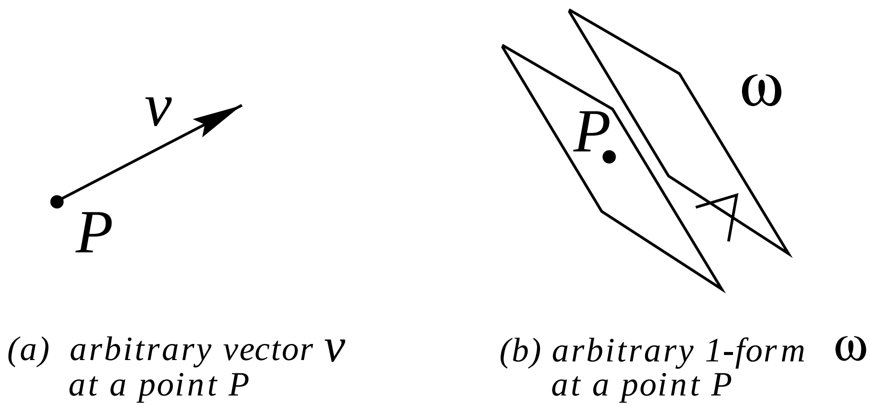

Figure A.2.6: (a) Image of a vector ('contravariant vector') at a point P. (b) Image of a 1-form ('covector' or 'covariant vector') at a point P.

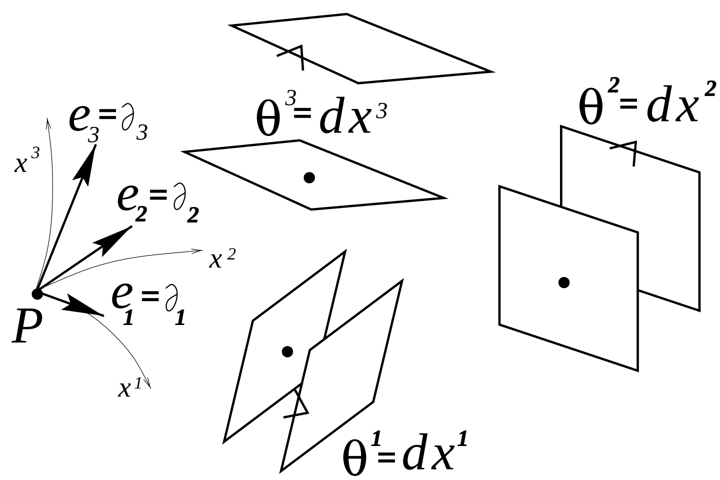

Figure A.2.7: Local coordinates (x^1, x^2, x^3) at a point P of a 3-dimensional manifold and the basis vectors (e_1, e_2, e_3). The basis 1-forms $\vartheta^i = dx^i$, $i = 1, 2, 3$, are supposed to be also at P. Note that $\vartheta^1(e_1) = 1$, $\vartheta^1(e_2) = 0$, $\vartheta^1(e_3) = 0$, etc., i.e. ϑ^i is dual to e_j according to $\vartheta^i(e_j) = \delta^i_j$.

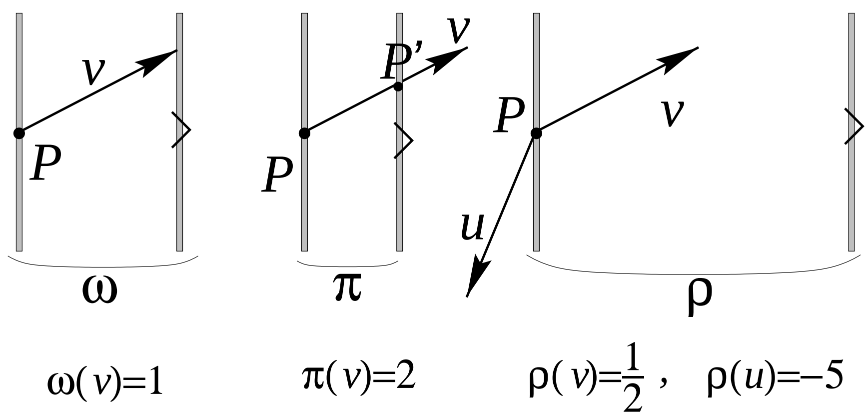

Figure A.2.8: Two-dimensional images of 1-forms ω, π, ρ of different strengths at P. They are applied to a certain vector v at the same point P and, in the case of ρ, also to the vector u.

To give a specific example, let us consider a three-dimensional manifold with local coordinates (x^1, x^2, x^3). A hyperplane is then simply a two-dimensional plane. A local basis of *vectors* is given by $e_i = \partial_i$, with $i = 1, 2, 3$, which are tangent to the coordinate lines x^i, respectively. Similarly, a local basis of *1-forms* $\vartheta^i = dx^i$ is represented by the local planes which depict surfaces of constant value of the relevant coordinate x^i, see Fig. A.2.7.

A 1-form is defined in such a way that, if applied to a vector, a number (scalar) pops out. The pictorial representation of such a number is straightforward, see Fig. A.2.8: A straight line starting at P in the direction of the vector v dissects the second hypersurface at P'. The number $\omega(v)$ is then the size of the vector v measured in terms of the segment PP' used as a unit.[1]

[1] Pictures of vectors and one-forms, and of many other geometrical quantities, can be found in Schouten [24], see, e.g., page 55. Also easily accessible is Misner et al. [18] where in Chapter 4 a number of corresponding worked out examples and nice pictures are displayed. More recent books with beautiful images of forms and their manipulation include those of Burke [2] and Jancewicz [12].

A.2.5 ⊗Volume forms and orientability

An everywhere non-vanishing n-form on an n-dimensional manifold is called a volume form.

Like in the case of a vector space V, differential forms of maximal rank on a manifold X are closely related to volume and orientation. Any *everywhere non-vanishing* n-form ω is called a *volume form* on X. Obviously, it is determined by a single component in every local coordinate chart (U_α, ϕ_α):

$$\omega = \frac{1}{n!}\,\omega_{i_1\ldots i_n}(x)\; dx^{i_1}\wedge\ldots\wedge dx^{i_n} = \omega_{1\ldots n}(x)\,dx^1\wedge\ldots\wedge dx^n. \tag{A.2.10}$$ volform

A manifold X is orientable *if and only if* it has a global volume form. Indeed, given a volume form ω which, by definition, does not vanish for any x, the function $\omega_{1\ldots n}(x)$ is either everywhere positive or everywhere negative in U_α. If it is negative, we can simply replace the local coordinate x^1 by $x^{1'} = -x^1$. Then $\omega_{1'2\ldots n}(x) > 0$. Thus, without any restriction, we have positive coefficient functions $\omega_{1\ldots n}(x)$ in all charts of the atlas $\{(U_\alpha, \phi_\alpha)\}$. In an intersection of any two charts $U_\alpha \bigcap U_\beta$ with local coordinates $\phi_\alpha = \{x^i\}$ and $\phi_\beta = \{y^i\}$, we have

$$\begin{aligned}\omega &= \omega_{1\ldots n}(y)\,dy^1\wedge\ldots\wedge dy^n = \omega_{1\ldots n}(x)\,dx^1\wedge\ldots\wedge dx^n\\ &= \omega_{1\ldots n}(x)\,J(f_{\alpha\beta})\,dy^1\wedge\ldots\wedge dy^n.\end{aligned} \tag{A.2.11}$$

Thus $\omega_{1\ldots n}(y) = \omega_{1\ldots n}(x)\,J(f_{\alpha\beta})$. Since both $\omega_{1\ldots n}(x)$ and $\omega_{1\ldots n}(y)$ are positive, we conclude that the Jacobian determinant $J(f_{\alpha\beta})$ is also positive for all intersecting charts, cf. (A.2.2). Hence the atlas $\{(U_\alpha, \phi_\alpha)\}$ is oriented.

Conversely, let the atlas $\{(U_\alpha, \phi_\alpha)\}$ be oriented, i.e., (A.2.2) holds true. Then in each chart (U_α, ϕ_α) we have an evidently non-vanishing n-form $\omega^{(\alpha)} := dx^1\wedge\ldots\wedge dx^n$. In the intersections $U_\alpha \bigcap U_\beta$ one finds $\omega^{(\alpha)} = J(f_{\alpha\beta})\,\omega^{(\beta)}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the covering $\{U_\alpha\}$ of X. Then we define a global n-form $\omega = \sum_\alpha \rho_\alpha\,\omega^{(\alpha)}$. Since in the overlapping charts all non-trivial forms $\omega^{(\alpha)}$ are positive multiples of each other and $\rho_\alpha(p) \geq 0$, $\sum_\alpha \rho_\alpha(p) = 1$ (i.e., all ρ_α cannot vanish at any point p), we conclude that ω is a volume form.

A.2.6 $^\otimes$Twisted forms

The differential forms that can be defined on a non-orientable manifold are called twisted differential forms. They are orientation-valued in terms of the conventional differential forms of Sec. A.2.3.

"Since the integral of a differential form on $\mathbb{R}^n$ is not invariant under the whole group of diffeomorphisms of $\mathbb{R}^n$, but only under the subgroup of orientation-preserving diffeomorphisms, a differential form cannot be integrated over a nonorientable manifold. However, by modifying a differential form we obtain something called a *density*, which can be integrated over any manifold, orientable or not."[2]

Besides the conventional differential forms, one can define slightly different objects which are called *twisted forms*[3] (or, sometimes, "odd forms", "impair forms", "densities" or "pseudoforms"). The twisted forms are necessary for an appropriate representation of certain physical quantities, such as the electric current density. Moreover, they are indispensable when one considers the integration theory on manifolds and, in particular, on non-orientable manifolds.

In Sec. A.1.3, see Examples 4) and 5), a twisted form was defined on a vector space V as a geometric quantity. Intuitively, a twisted form on the manifold X can be defined as an "orientation-valued" conventional exterior form. Given an atlas $\{(U_\alpha, \phi_\alpha)\}$, a *twisted p-form* is represented by a family of differential p-forms

[2] Bott & Tu [1] p.79.

[3] "Twisted tensors were introduced by Hermann Weyl....and de Rham... called them tensors of odd kind... We could make a good case that the usual differential forms are actually the twisted ones, but the language is forced on us by history. Twisted differential forms are the natural representations for densities, and sometimes are actually called densities, which would be an ideal name were it not already in use in tensor analysis. I agonized over a notation for twisted tensors, say, a different typeface. In the end I decided against it..." William G. Burke [2], p. 183.

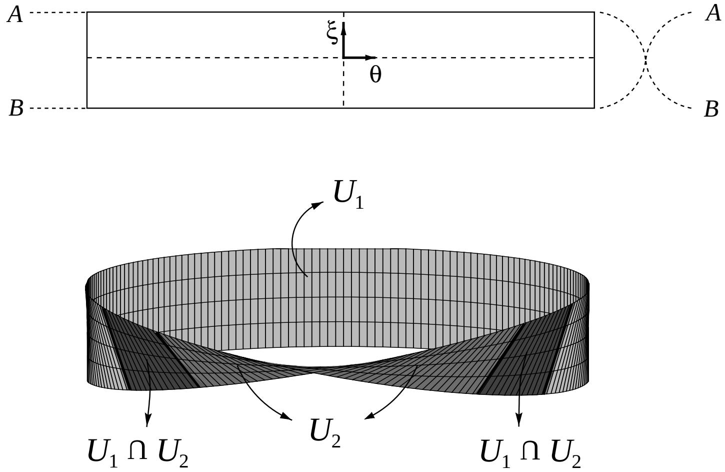

Figure A.2.9: Möbius strip.

$\{\omega^{(\alpha)}\}$ such that in the intersections $U_\alpha \bigcap U_\beta$

$$\omega^{(\alpha)} = \operatorname{sgn} J(f_{\alpha\beta})\, \omega^{(\beta)}. \qquad \text{(A.2.12)}$$ twist

Example: Consider the Möbius strip, a non-orientable two-dimensional compact manifold with boundary, see Fig. A.2.9. It can be easily realized by taking a rectangle $\{(\theta, \xi) \in \mathbb{R}^2 \,|\, 0 < \theta < 2\pi,\ -1 < \xi < 1\}$ and gluing it together with one twist along vertical sides. The simplest atlas for the resulting manifold consists of two charts (U_1, ϕ_1) and (U_2, ϕ_2). The open domains $U_{1,2}$ are rectangles and they can be chosen as shown in Fig. A.2.9, with the evidently defined local coordinate maps $\phi_1 = (x^1, x^2)$ and $\phi_2 = (y^1, y^2)$, where the first coordinate runs along the rectangles and the second one across them. The intersection $U_1 \bigcap U_2$ is comprised of two open sets, $(U_1 \bigcap U_2)_{\text{left}}$ and $(U_1 \bigcap U_2)_{\text{right}}$. The transition functions $f_{12} = \phi_1 \circ \phi_2^{-1}$ are $f_{12} = \{x^1 = y^1, x^2 = y^2\}$ in $(U_1 \bigcap U_2)_{\text{left}}$ and $f_{12} = \{x^1 = y^1, x^2 = -y^2\}$ in $(U_1 \bigcap U_2)_{\text{right}}$, so that $J(f_{12}) = \pm 1$ in these domains, respectively. The 1-form $\omega = \{\omega^{(1)} = dx^2, \omega^{(2)} = dx^2\}$ is a twisted form on the Möbius strip.

In general, given a chart (U_α, ϕ_α), both a usual and a twisted p-form is given by its components $\omega_{i_1 \dots i_p}(x)$, see (A.2.9). With a change of coordinates, the components of a twisted form, via (A.2.12), are transformed as

$$\omega_{i_1\ldots i_p}(x) = \left(\text{sgn}\ \det \frac{\partial x^i}{\partial y^j}\right) \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_p}}{\partial x^{i_p}}\ \omega_{j_1\ldots j_p}(y). \quad \text{(A.2.13)}$$

For a conventional p-form, the first factor on the right-hand side, the sign of the Jacobian, is absent.

Normally, in gravity and in field theory one works on orientable manifolds with an oriented atlas chosen. Then the difference between ordinary and twisted objects disappears, because of (A.2.2). However, twisted forms are very important on non-orientable manifolds on which usual forms cannot be integrated.

A.2.7 Exterior derivative

The exterior derivative maps a p-form into a $(p+1)$-form. Its crucial property is nilpotency, $d^2 = 0$.

Denote the set of vector fields on X by X_0^1. For 0-forms $f \in \Lambda^0(X)$ the differential 1-form df is defined by (A.2.7), (A.2.8), i.e., by $df = f_{,i}\, dx^i$. We wish to extend this map $d : \Lambda^0(X) \longrightarrow \Lambda^1(X)$ to a map $d : \Lambda^p(X) \longrightarrow \Lambda^{p+1}(X)$. Ideally this should be performed in a coordinate-free way and we shall give such a definition at the end of this section. However, the definition of exterior derivative of a p-form in terms of a coordinate basis is very transparent. Furthermore, it is a simple matter to prove that it is, in fact, independent of the local coordinate system that is used. Starting with the expression (A.2.9) for a $\omega \in \Lambda^p(X)$, we define $d\omega \in \Lambda^{p+1}(X)$ by

$$d\,\omega := \frac{1}{p!}\, d\,\omega_{i_1\ldots i_p} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}\,. \qquad \text{(A.2.14)} \quad \texttt{exdf}$$

By (A.2.7), (A.2.8), the right-hand side of (A.2.14) is

$$(1/p!)\,\omega_{i_1\ldots i_p,j}\ dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}\,. \qquad \text{(A.2.15)}$$

Hence, because of the antisymmetry of the exterior product, we may write

$$\boxed{d\,\omega = \frac{1}{p!}\,\omega_{[i_1\ldots i_p,j]}\;dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}\,.} \tag{A.2.16}$$

domega

Under a coordinate transformation $\{x^i\} \to \{x^{i'}\}$ it is found that

$$\omega_{[i'\ldots i'_p,j']} = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \cdots \frac{\partial x^{i_{p+1}}}{\partial x^{j'}}\;\omega_{[i_1\ldots i_p,i_{p+1}]}\,. \tag{A.2.17}$$

Hence,

$$\begin{aligned}\omega_{[i'_1\ldots i'_p,j']}\;dx^{j'} \wedge dx^{i'_1} \wedge \ldots \wedge dx^{i'_p} &= \\ = \omega_{[i_1\ldots i_p,i_{p+1}]}\;dx^{i_{p+1}} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}\,,\end{aligned} \tag{A.2.18}$$

so that the exterior derivative, as defined by (A.2.16), is independent of the coordinate system chosen.

Proposition 1: The exterior derivative, as defined by (A.2.16), is a map

$$d:\;\Lambda^p(X) \longrightarrow \Lambda^{p+1}(X) \tag{A.2.19}$$

d

with the following properties:

1) $d(\omega + \lambda) = d\omega + d\lambda$ [linearity],

2) $d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^p\,\omega \wedge d\phi$ [(anti-)Leibniz rule],

3) $df(u) = u(f)$ [partial derivative for functions],

4) $d(d\omega) = 0$ [nilpotency].

Here, $\omega,\,\lambda \in \Lambda^p(X), \phi \in \Lambda^q(X), f \in \Lambda^0(X), u \in X^1_0(X)$.

Proof. 1) and 3) are obvious from the definition.
Because of 1) and the distributive property of the exterior multiplication, it is sufficient to prove 2) for ω and ϕ of the 'monomial' form:

$$\omega = f\,dx^{i_1} \wedge \ldots \wedge dx^{i_p}\,, \qquad \phi = h\,dx^{j_1} \wedge \ldots \wedge dx^{j_q}\,. \tag{A.2.20}$$

monom

Thus

$$\omega \wedge \phi = f h \, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q} \, . \tag{A.2.21}$$

monom2

Then

$$\begin{aligned} d(\omega \wedge \phi) &= d(fh) \wedge dx^{i_1} \wedge \ldots \wedge dx^{j_q} \\ &= (f \, dh + h \, df) \wedge dx^{i_1} \wedge \ldots \wedge dx^{j_q} \\ &= (df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \wedge (h \, dx^{j_1} \wedge \ldots \wedge dx^{j_q}) \\ &\quad + (-1)^p \, (f \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \wedge (dh \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}) \\ &= d\omega \wedge \phi + (-1)^p \, \omega \wedge d\phi \, . \end{aligned} \tag{A.2.22}$$

To prove 4), we first of all note that, for a function $f \in \Lambda^0(X)$,

$$d(df) = f_{,[ij]} \, dx^j \wedge dx^i = 0 \, , \tag{A.2.23}$$

ddo

since partial derivatives commute. For a p-form it is sufficient to consider a monomial

$$\omega = f \, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \, . \tag{A.2.24}$$

mono

Then

$$d\omega = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \tag{A.2.25}$$

and repeated application of property 2) and (A.2.23) yields the desired result $d(d\omega) = 0$. By linearity, this may be extended to a general p-form which is a linear combination of terms like (A.2.24).

Proposition 2: Invariant expression for the exterior derivative. For $\omega \in \Lambda^p(X)$, we can express $d\omega$ in a coordinate-free manner as follows:

$$\begin{aligned} d\,\omega(u_0, u_1, \ldots, u_p) &= \sum_{j=0}^{p} (-1)^j \, u_j \left(\omega(u_0, \ldots, \widehat{u}_j, \ldots, u_p) \right) \\ &\quad + \sum_{0 \le i < j \le p} (-1)^{i+j} \, \omega([u_i, u_j], u_0, \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots, u_p) \, , \end{aligned} \tag{A.2.26}$$

coordfree

where $u_0, u_1, \ldots, u_p$ are arbitrary vector fields and $\widehat{u}$ indicates that the field u is *omitted* as an argument. It is a straightforward matter to verify that (A.2.26) is consistent with (A.2.16). We shall make particular use of the case in which ω is a 1-form and (A.2.9) becomes

$$d\omega(u,v) = u\big(\omega(v)\big) - v\big(\omega(u)\big) - \omega([u,v])\,. \qquad \text{(A.2.27)}$$ doneform

A.2.8 Frame and coframe

A natural frame and natural coframe are defined at every local coordinate patch by ∂_i and dx^i, respectively. An arbitrary frame e_α and coframe ϑ^α are constructed by a linear transformation therefrom. Anholonomity object measures of how much is a coframe different from a natural one.

A *local* frame on an n-dimensional differentiable manifold X is a set e_α, $\alpha = 0, 1, \ldots, n$, of n vector fields that are *linearly independent* at each point of an open subset U of X. They thus form a basis of the tangent (vector) space X_x at every point $x \in U$. There exist quite ordinary manifolds, the 2-dimensional sphere, for example, where no continuous frame field can be introduced globally, i.e., at each point of the manifold X. Therefore, speaking of frames on X, we will always have in mind local frames. If e_α is a frame, the corresponding *coframe* is the set ϑ^α of n different 1-forms such that

$$\vartheta^\alpha(e_\beta) = \delta^\alpha_\beta \qquad \text{(A.2.28)}$$ defcoframe

is valid at each point of X. In other words $\vartheta^\alpha|_x$ at each point $x \in X$ is the dual basis of 1-forms for X^*_x. We note in particular that, as a consequence of (A.2.28), every vector field $u \in X^1_0$ can be decomposed according to

$$u = u^\alpha\, e_\alpha, \quad \text{where} \quad u^\alpha = \vartheta^\alpha(u) = u \rfloor \vartheta^\alpha. \qquad \text{(A.2.29)}$$ vecdec

A local *coordinate* system defines a coordinate frame ∂_i on the open neighborhood U. Thus an arbitrary frame e_α may be expressed on U in terms of ∂_i in the form of

$$e_\alpha = e^i{}_\alpha \; \partial_i \,, \tag{A.2.30}$$
eia

where $e^i{}_\alpha$ are differentiable functions of the coordinates. For the corresponding coframe ϑ^α we have

$$\vartheta^\alpha = e_i{}^\alpha \, dx^i \,, \tag{A.2.31}$$
eai

where, by (A.2.28),

$$e_i{}^\alpha \, e^i{}_\beta = \delta^\alpha_\beta \,. \tag{A.2.32}$$

Provided a coframe ϑ^α has the property that

$$d\vartheta^\alpha = 0 \,, \tag{A.2.33}$$

it is said to be *natural* or *holonomic*. In this case, in the neighborhood of each point, there exists a coordinate system $\{x^i\}$ such that

$$\vartheta^\alpha = \delta^\alpha_i dx^i \,. \tag{A.2.34}$$
holframe

Under these circumstances, also the frame e_α is natural or holonomic with $e_\alpha = \delta^i_\alpha \, \partial_i$. The 2-form

$$C^\alpha := d\vartheta^\alpha = \frac{1}{2} \, C_{ij}{}^\alpha \, dx^i \wedge dx^j = \frac{1}{2} \, C_{\beta\gamma}{}^\alpha \, \vartheta^\beta \wedge \vartheta^\gamma \,, \tag{A.2.35}$$
nonholobj

with $C_{(\beta\gamma)}{}^\alpha \equiv 0$, is the object of *anholonomity* with its 24 independent components. It measures how much a given coframe ϑ^α fails to be holonomic. There is also a version of (A.2.35) in terms of the frame e_α. With the help of (A.2.27), it can be rewritten as

$$[e_\alpha, e_\beta] = - \, C_{\alpha\beta}{}^\gamma \, e_\gamma \,. \tag{A.2.36}$$
commut

The object of anholonomity has a non-tensorial transformation behavior.

A.2.9 $^{\otimes}$Maps of manifolds: push-forward and pull-back

Pull-back φ^* and push-forward φ_* maps are the companions of every diffeomorphism φ of the manifold X. They relate the corresponding cotangent and tangent spaces at points x and $\varphi(x)$. Both maps commute with the exterior differential.

If a differentiable map $\varphi : X \to Y$ is given, various geometric objects can be transported either from X to Y (pushed forward) or from Y to X (pulled back). A push-forward will be denoted by φ_* and a pull-back by φ^*.

Given a tangent vector u at a point $x \in X$, we can define its push-forward $\varphi_* u \in Y_{\varphi(x)}$ (which is also called *differential*) by determining its action on a function $f \in C(Y)$ as

$$(\varphi_* u)\,(f) = u(f \circ \varphi). \tag{A.2.37}$$ pushf

However, if u is not merely a tangent vector, but a vector field over X, it is in general not possible to define its push-forward to Y. There might be two reasons for that. Firstly, if φ is not injective and $\varphi(x_1) = \varphi(x_2)$ for $x_1 \neq x_2$, then the vectors pushed from X_{x_1} and X_{x_2} will be different in general. Secondly, if φ is not surjective, the push-forwarded vector field would not, in general, be determined all over Y. It is always possible to define $\varphi_* v$ of a vector field if φ is a *diffeomorphism* (which can only be considered when $\dim X = \dim Y$).

Using the rule

$$\varphi_*(v_1 \otimes v_2) := \varphi_* v_1 \otimes \varphi_* v_2\,, \tag{A.2.38}$$ pushf2

we can define the push-forward of an arbitrary contravariant tensor at $x \in X$ to the space of tensors of the same type at $\varphi(x) \in Y$. So φ_* becomes a homomorphism of the algebras of contravariant tensors at $x \in X$ and $\varphi(x) \in Y$.

In a diagram we can depict the push-forward map φ_* of tangent vectors u and the pull-back map φ^* for 1-forms ω:

$$
\begin{array}{ccc}
\varphi^*\omega \in X^*_x & \xleftarrow{\ \varphi^*\ } & \omega \in Y^*_{\varphi(x)} \\
\uparrow & & \uparrow \\
x \in X & \xrightarrow{\ \varphi\ } & y = \varphi(x) \in Y \\
\downarrow & & \downarrow \\
u \in X_x & \xrightarrow[\ \varphi_*\ ]{} & \varphi_* u \in Y_{\varphi(x)}
\end{array}
$$

Let $\{x^i\}$ be local coordinates in X and $\{y^j\}$ the local coordinates in Y (with the ranges of indices i and j defined by the dimensionality of X and Y, respectively). Then the map φ is described by a set of smooth functions $y^j(x^i)$, and the push-forward map for the tensors of type $\left[{p \atop 0}\right]$ in components read

$$(\varphi_* T)^{j_1 \dots j_p} = \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_p}}{\partial x^{i_p}}\bigg|_x T^{i_1 \dots i_p}. \qquad \text{(A.2.39)}$$ locpush

Comparing with (A.2.8) for the case when $Y = \mathbb{R}$, it becomes clear why φ_* is also called a differential map.

For a p-form $\omega \in \Lambda^p_{y=\varphi(x)}(Y)$, we can determine its pull-back $\varphi^*\omega \in \Lambda^p_x(X)$ by

$$(\varphi^*\omega)\,(v_1, \dots, v_p) = \omega(\varphi_* v_1, \dots, \varphi_* v_p)\,. \qquad \text{(A.2.40)}$$ pullb

This definition can be straightforwardly extended to a homomorphism of the algebras of covariant type $\left[{0 \atop p}\right]$ tensors. In local coordinates it reads, analogously to (A.2.39),

$$(\varphi_* T)_{i_1 \dots i_p} = \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_p}}{\partial x^{i_p}}\bigg|_x T_{j_1 \dots j_p}. \qquad \text{(A.2.41)}$$ locpull

Let ω be an exterior p-form (i.e., a p-form *field*) on Y. In order to determine its pull-back $\varphi^*\omega$ to X by (A.2.40), it is sufficient to have $\varphi_* v_1, \dots, \varphi_* v_p$ on the right hand side of (A.2.40) defined as vectors (i.e., not necessarily as vector fields). Therefore, the pull-back of exterior forms (and, in general, of contravariant vector fields) is determined for an arbitrary map φ. In exterior calculus, an important property is the commutativity of pull-back and exterior differentiation for any p-form ω:

$$\boxed{d(\varphi^*\omega) = \varphi^*(d\omega)\,.} \tag{A.2.42}$$

dpull

If φ is a *diffeomorphism*, or at least a local diffeomorphism, we shall use the pull-back φ^* of arbitrary tensor fields. For contravariant tensors, it can be defined as

$$\varphi^* = (\varphi^{-1})_* = (\varphi_*)^{-1}\,. \tag{A.2.43}$$

pullcontra

To define it for an arbitrary tensor of type $\left[{p \atop q}\right]$, we have to require only that φ^* is an algebra isomorphism. Technically, in local coordinates, this amounts to the invertibility of the square matrices $\partial y^j/\partial x^i$.

When φ is a (local) diffeomorphism, we can also pull-back (or push-forward) geometric quantities constructed in tangent space. Let $[(w,e)]$ be a geometric quantity; here $e = (e_1,\ldots,e_n)$ is a frame in the tangent space Y_y and w belongs to the set W, in which there is the left action ρ of $GL(n,\mathbb{R})$. Like for vectors, we define $\varphi^* e = \big(\varphi^* e_1,\ldots,\varphi^* e_n\big)$ and

$$\varphi^*[(w,e)] = [(w,\varphi^* e)]\,, \tag{A.2.44}$$

pullrho

i.e., the transported object has the same components as the initial object with respect to the transported frame. Certainly, this definition of the pull-back is consistent with that given earlier for tensors.

A.2.10 ⊗Lie derivative

> *A vector field generates a group of diffeomorphisms on a manifold. Making use of this group action, the Lie derivative enables us to compare tensors and geometric quantities at different points.*

The main result of the present section will be equation (A.2.51) on the Lie derivative of a differential form. However, we shall first explain the concept of Lie derivative of a general geometrical quantity. Note that for a Lie derivative *no metric and no connection is required*, it can be defined on each differentiable manifold.

For each point $p \in X$, a vector field u, with $u(p) \neq 0$, determines a unique curve $\sigma_p(t)$, $t \geq 0$ such that $\sigma_p(0) = p$ with u as the tangent vector field to the curve. The family of curves defined in this way is called the *congruence* of curves generated by the vector field u. Let $\{x^i\}$ be a local coordinate system with x^i_p as coordinates of p and decompose u according to $u = u^i(x^1, \dots, x^n)\, \partial_i$. Then the curve $\sigma_p(t)$ is found by solving the system of ordinary differential equations

$$\frac{dx^i}{dt} = u^i\big(x^1(t), \dots , x^n(t)\big)\,, \tag{A.2.45}$$

dsystem

with initial values $x^i(0) = x^i_p$. The congruence of curves obtained in this way defines (at least locally) a 1-parameter group of diffeomorphisms φ_t on X given by

$$\varphi_t(p) = \sigma_p(t), \qquad \forall p \in X\,, \tag{A.2.46}$$

with the properties that (a) $\varphi_t^{-1} = \varphi_{-t}$, (b) $\varphi_t \circ \varphi_s = \varphi_{t+s}$, and (c) φ_0 is the identity map. The integral curves of the congruence are called the *trajectories* of the group. Furthermore, the equations (A.2.45) are equivalent to

$$u(f)\,(p) = \lim_{t\to 0} \frac{f\,(\varphi_t(p)) - f\,(p)}{t} = \frac{d}{dt}\, f\Big(\varphi_t(p)\Big)\Big|_{t\,=\,0}\,, \tag{A.2.47}$$

uf

for all $p \in X$ and all differentiable functions f.

Examples in $\mathbb{R}^2$:

1) The vector field $u = \partial/\partial x$ generates translations $\varphi_t(x, y) = (x + t, y)$, $-\infty < t < +\infty$. The trajectories are the lines $y =$ constant. See Fig. A.2.10a.

2) The vector field $u = (x\,\partial/\partial y - y\,\partial/\partial x)$ generates the circular motion $\varphi_t(x, y) = (x \cos t - y \sin t,\ x \sin t + y \cos t)$, $0 \leq t < 2\pi$. The trajectories are concentric closed curves around the origin, see Fig. A.2.10b.

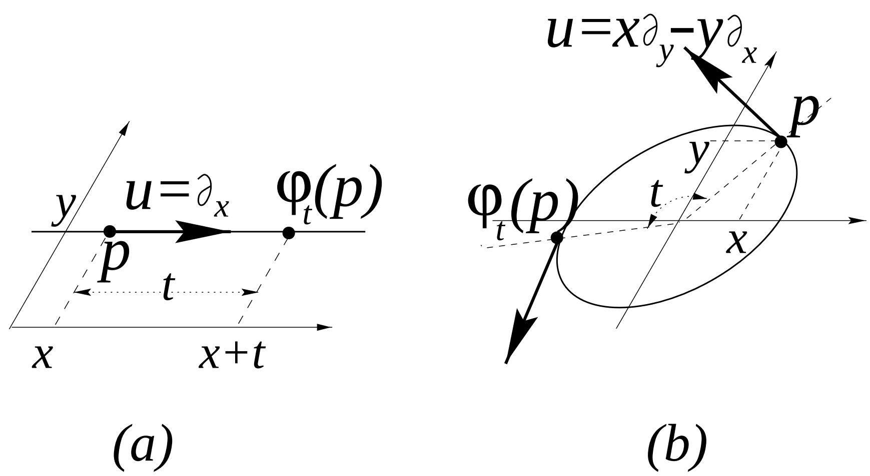

Figure A.2.10: Translations (a) and circular motion (b) generated on $\mathbb{R}^2$.

In general, if we take a coordinate patch U of a differentiable manifold with coordinates $\{x^i\}$, then φ_t is defined in terms of x^i by

$$\varphi_t\,(x^i) = y^i := f^i(t;\;x^j)\,, \tag{A.2.48}$$

where $f^i(t;\,x^j)$ are differentiable functions of (t, x^j). Property (a) states that $x^i = x^i(t; y^j) = f^i(-t; y^j)$. By property (b) we have $f^i\left(t; f^j(s; x^k)\right) = f^i(t+s; x^j)$ while (c) means that $f^i(0; x^j) = x^i$.

For every value of t in a certain interval, the diffeomorphism φ_t induces corresponding pull-backs φ_t^* on functions, vectors, exterior forms, and general tensor fields of type $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$. Accordingly, the *Lie derivative* of a tensor T with respect to a vector field u is defined by

$$\boxed{\mathcal{L}_u T := \lim_{t\to 0}\frac{\varphi_t^* T - T}{t}\,.} \tag{A.2.49}$$ Liedef

It is sufficient to have explicit expressions for the Lie derivatives of functions, vectors, and 1-forms, in order be in a position to do the same for a general tensor. The two most important cases are as follows:

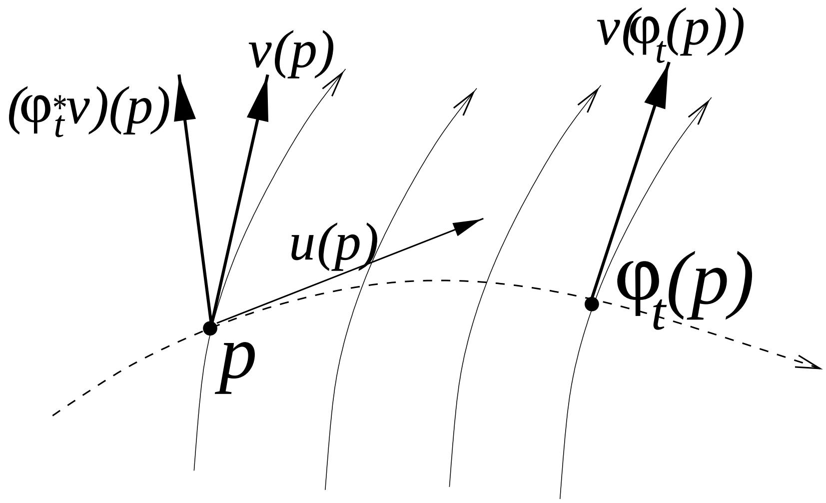

Figure A.2.11: To the definition of Lie derivative $\mathcal{L}_u v$ with respect to a vector u: The one-parameter group φ_t, generated by the vector field u, is used in order to transfer the vector $v(\varphi_t(p))$ back to the initial point and to compare it with $v(p)$.

For vectors $v \in X^1_0$:

$$\boxed{\mathcal{L}_u v = [u, v],} \tag{A.2.50}$$ liev

see (A.2.6) for a component version. For p-forms $\omega \in \Lambda^p(X)$ and $p \geq 0$, we find the *main theorem* for the Lie derivative of an exterior form:

$$\boxed{\mathcal{L}_u \omega = u \rfloor (d\omega) + d(u \rfloor \omega).} \tag{A.2.51}$$ lietheorem

An alternative coordinate-free general formula for this Lie derivative reads:

$$\begin{aligned}(\mathcal{L}_u \omega)(v_1, \ldots, v_p) =& u\left(\omega(v_1, \ldots, v_p)\right) \\ & - \sum_{i=1}^{p} \omega(v_1, \ldots, [u, v_i], \ldots, v_p).\end{aligned} \tag{A.2.52}$$ Lieinv

The Lie derivative for the functions $f \in C(X)$ is obtained as a particular case of (A.2.51) for $p = 0$:

$$\mathcal{L}_u f = u(f) = u \rfloor df. \tag{A.2.53}$$ lief

The last formula is straightforwardly checked by a direct calculation,

$$\begin{aligned}\mathcal{L}_u f\,(p) &= \lim_{t\to 0}\frac{(\varphi_t^* f)\,(p) - f(p)}{t}\\ &= \lim_{t\to 0}\frac{f(\varphi_t(p)) - f(p)}{t} = u(f)(p)\,,\end{aligned} \qquad (A.2.54)$$ liefunction

by use of (A.2.47). The proof of (A.2.50) and (A.2.51) is left as an exercise to the readers. As a hint, we mention that the formula (A.2.51) follows from the Lie derivative of a 1-form ω,

$$(\mathcal{L}_u\omega)\,(v) = u\,(\omega(v)) - \omega\,(\mathcal{L}_u v)\,, \qquad (A.2.55)$$ lieform

where u, v are arbitrary vector fields.

The most important properties of the *Lie derivatives of exterior forms* may be summarized as follows:

1) $\mathcal{L}_u d\omega = d\mathcal{L}_u\omega$ [$\mathcal{L}$ and d commute],

2) $\mathcal{L}_u(\omega\wedge\varphi) = (\mathcal{L}_u\omega)\wedge\varphi + \omega\wedge\mathcal{L}_u\varphi$ [Leibniz rule],

3) $\mathcal{L}_{fu}\omega = f\mathcal{L}_u\omega + df\wedge(u\,\lrcorner\,\omega)$ [rescaled vector],

4) $\mathcal{L}_v\mathcal{L}_u\omega - \mathcal{L}_u\mathcal{L}_v\omega = \mathcal{L}_{[u,v]}\omega$ [non-commutativity],

5) $\mathcal{L}_v(u\,\lrcorner\,\omega) - u\,\lrcorner\,\mathcal{L}_v\omega = [v,u]\,\lrcorner\,\omega$ [$\mathcal{L}$ and $\lrcorner$ do not commute].

The formulas above contain all necessary information about the Lie derivative for arbitrary tensors of type $\left[{p \atop q}\right]$. In particular, by construction we have that $\mathcal{L}_u$ is type preserving, i.e. if $T\in T^p_q(X)$ then $(\mathcal{L}_u T)\in T^p_q(X)$. Moreover, it is clear that, for any two tensor fields T and S of the same type,

$$\mathcal{L}_u(T+S) = \mathcal{L}_u T + \mathcal{L}_u S \qquad (A.2.56)$$ liesum

and

$$\mathcal{L}_{u+v}T = \mathcal{L}_u T + \mathcal{L}_v T\,. \qquad (A.2.57)$$ liesum1

Finally, for $T \in T^r_s(X)\,, S \in T^m_q(X)$,

$$\mathcal{L}_u(T \otimes S) = (\mathcal{L}_u T) \otimes S + T \otimes (\mathcal{L}_u S)\,. \qquad \text{(A.2.58)}$$ lieleibniz

The proof follows straightforwardly from

$$\varphi_t^*(T \otimes S) = \varphi_t^* T \otimes \varphi_t^* S\,. \qquad \text{(A.2.59)}$$

These properties enable us to express the Lie derivative of a general tensor in terms of a local coordinate basis. Consider a tensor field of type $\left[{2 \atop 1}\right]$, for example. In terms of a local coordinate system $\{x^i\}$,

$$T = T^{ij}{}_k(x)\,\partial_i \otimes \partial_j \otimes dx^k\,, \qquad \text{(A.2.60)}$$

and we easily find for $u = \xi^i(x)\,\partial_i$:

$$\mathcal{L}_u T = \left(T^{ij}{}_{k,r}\,\xi^r - T^{rj}{}_k\,\xi^i{}_{,r} - T^{ir}{}_k\,\xi^j{}_{,r} + T^{ij}{}_r\,\xi^r{}_{,k}\right)\,\partial_i \otimes \partial_j \otimes dx^k\,. \qquad \text{(A.2.61)}$$

For completeness, let us consider the Lie derivative of a *geometric quantity*. For this purpose we note the following: If $e_\alpha(x)$ is a frame taken at a given point $p \in X$, then $\varphi_t^*\big(e_\alpha(\varphi_t(x))\big)$ can be decomposed with respect to this frame with some t-dependent coefficients:

$$\varphi_t^*\big(e_\alpha(\varphi_t(x))\big) = \Phi_\alpha{}^\beta(-t,x)\,e_\beta(x). \qquad \text{(A.2.62)}$$

Differentiating this equation with respect to t at $t = 0$, we get

$$[u, e_\alpha] = -\frac{d}{dt}\Phi_\alpha{}^\beta(t,x)\Big|_{t\,=\,0}\,e_\beta(x)\,, \qquad \text{(A.2.63)}$$

and thus formally we find the matrix

$$\psi_\alpha{}^\beta := \frac{d}{dt}\Phi_\alpha{}^\beta(t,x)\Big|_{t\,=\,0} = e_\alpha(u^\beta)(x) + u^\mu(x)\,C_{\mu\alpha}{}^\beta(x)\,. \qquad \text{(A.2.64)}$$ psiab

Let us consider a field of a geometric quantity $[(w(x), e(x))]$ of type ρ, for short $w(x)$. According to the definition of the pull-back of such objects, we have

$$\begin{aligned}\left(\varphi_t^*[(w, e)]\right)(x) &= \left[\left(w(\varphi_t(x)),\ \varphi_t^*(e(\varphi_t(x)))\right)\right] \\ &= \left[\left(w(\varphi_t(x)), \Phi(-t, x)\, e(x)\right)\right] \\ &= \left[\left(\rho(\Phi(t, x))\, w(\varphi_t(x)),\ e(x)\right)\right] .\end{aligned} \tag{A.2.65}$$

We differentiate at $t = 0$ and find:

$$\begin{aligned}\left(\mathcal{L}_u w\right)(x) &= \frac{d}{dt}\left(\rho(\Phi(t, x))\, w(\varphi_t(x))\right)\Big|_{t\,=\,0} \\ &= (uw)(x) + \rho_* \psi(x)\, w(x) .\end{aligned} \tag{A.2.66}$$

lierho

In all practical applications, a geometric quantity is described as a smooth field on X which takes values in the vector space $W = \mathbb{R}^N$ of a ρ-representation of the group $GL(n, \mathbb{R})$ of local linear frame transformations. In simple terms, the geometric quantity $w = w^A\, e_A$ is represented by its components w^A with respect to the basis e_A of the vector space $W = \mathbb{R}^N$. Hereafter $A, B, \dots = 1, \dots, N$. Thus, recalling (A.1.17), $\rho(L) = \rho_A{}^B(L) \in GL(N, \mathbb{R})$, and ρ_* maps the tangent space $End(n, \mathbb{R})$ of the group $GL(n, \mathbb{R})$ into the tangent space $End(N, \mathbb{R})$ of the group $GL(N, \mathbb{R})$ by means of the matrix

$$\rho_A{}^{B\alpha}{}_\beta := \frac{\partial \rho_A{}^B(L)}{\partial L_\alpha{}^\beta}\Bigg|_{L_\alpha{}^\beta\,=\,\delta_\alpha^\beta} . \tag{A.2.67}$$

rhoAB

Therefore, taking into account (A.1.18), Eq.(A.2.66) reads for the components of the geometric quantity:

$$\mathcal{L}_u w^A = u(w^A) - \rho_B{}^{A\alpha}{}_\beta\, \psi_\alpha{}^\beta\, w^B , \tag{A.2.68}$$

where $\psi_\alpha{}^\beta$ is defined by (A.2.64).

For a tensor field $T^\alpha{}_\beta$ of type $\left[{1 \atop 1}\right]$, for instance, we have

$$\mathcal{L}_u T^\alpha{}_\beta = u\left(T^\alpha{}_\beta\right) + \psi_\beta{}^\mu\, T^\alpha{}_\mu - \psi_\mu{}^\alpha\, T^\mu{}_\beta . \tag{A.2.69}$$

lierho1

As an interesting exercise, we propose to the reader to calculate the Lie derivative of a scalar density $\mathcal{S}$ of weight w, see (A.1.57):

$$\mathcal{L}_u \mathcal{S} = u(\mathcal{S}) + w \mathcal{S} \psi_\alpha{}^\alpha . \tag{A.2.70}$$

liedens

It is a simple matter to generalize the relation for the Lie derivative of a p-form of type ρ. If ω is such a form, then

$$\mathcal{L}_u\omega = d(u \rfloor \omega) + u \rfloor d\omega + \rho(\psi)\,\omega\,. \tag{A.2.71}$$ lierho2

For a vector-valued p-form, for example, we have

$$\mathcal{L}_u\omega^\alpha = d(u \rfloor \omega^\alpha) + u \rfloor d\omega^\alpha - \omega^\beta \psi_\beta{}^\alpha\,. \tag{A.2.72}$$ lierho3

A.2.11 Excalc, a Reduce package

In this Chap. A.2, we introduced successively, after an n-dimensional manifold had been specified, fields of 1-forms, p-forms, and vectors. Their declarations by means of `pform` and `tvector` are already known to us. Then the exterior derivative was specified. In Excalc, not surprizingly, the letter `d` is reserved for this operator. Partial differentiation is denoted by the operator @. Thus, `@(sin x,x);` yields `cos(x)`. We collect the different Excalc operators in Table A.2.1.

Math.	Excalc	Operator	Operator Type
$\wedge$	`^`	exterior product	nary infix
$\rfloor$	`_|`	interior product	binary infix
∂	`@`	partial derivative	nary prefix
d	`d`	exterior derivative	unary prefix
£	`|_`	Lie derivative	binary infix
$\star$	`#`	Hodge star operator	unary prefix

Table A.2.1: Translation of mathematical symbols into Excalc. The Hodge star operator will not be defined before Sec. C.2.8.

Unary means that there is one, binary that there are two arguments, and "nary" means that there is any number of arguments.

Let us load again Excalc by `load_package excalc$`. By means of a declaration with `fdomain`, an identifier can be declared to be a function of certain variables. With

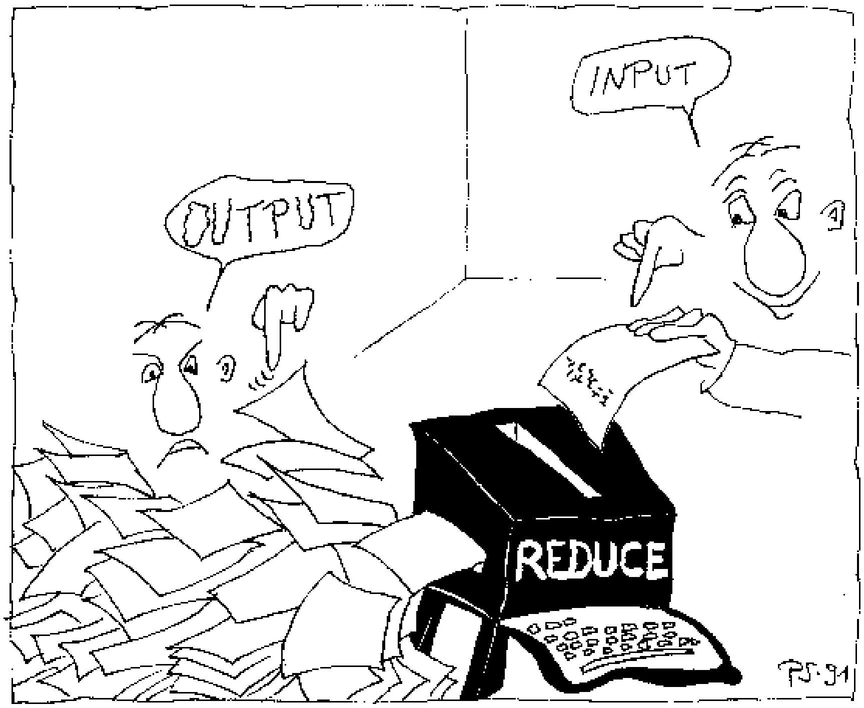

Figure A.2.12: The perennial computer algebra problem.

```
fdomain f=f(x,y),h=h(x);
@(x*f,x);   @(h,y);
```

we find, respectively,

```
f + x*@  f
       x


0
```

The partial derivative symbol can also be an operator with a single argument, like in `@(z)`. Then it represents the leg ∂_z of a natural frame.

Coming back to the exterior derivative, the following example is now self-explanatory:

```
pform x=0,y=0,z=0,f=0;
```

```
fdomain f=f(x,y);
d f;

@  f*d x + @  f*d y
 x          y
```

Products are normally differentiated out, i.e.

```
pform x=0,y=p,z=q;
d(x*y^z);

      p
( - 1) *x*y^d z  + x*d y^z + d x^y^z
```

This expansion can be suppressed by the command `noxpnd d;` Expansion is performed again when the command `xpnd d;` is executed.

The Excalc operator `d` knows all the rules for the exterior derivative as specified in Proposition 1 in the context of (A.2.19). Let us declare the corresponding ranks of the forms in order to check the first two rules (note that *lambda* is a reserved identifier in Reduce and cannot be used):

```
pform omega=p, lam=p, phi=q;
```

Then we give the commands

```
d(omega+lam);  d(omega^phi);
```

and find, respectively

```
d lam + d omega

      p
( - 1) *omega^d phi + d omega^phi
```

The last but one entry in our table is the Lie derivative £. In Excalc, it can be applied to an exterior form with respect to a vector or to a vector again with respect to a vector. It is represented by the infix operator |_ (vertical bar followed by an underline). If the Lie derivative is applied to a form, Excalc remembers the main theorem of Lie derivatives, namely (A.2.51). Thus,

```
pform z=k;  tvector u;  u|_z;
```

yields

```
d(u _| z) + u _| d z
```

The operator of the Lie derivatives fulfills the rules displayed after (A.2.55). We will check the rule for the rescaled vector as an example. Already above, the form ω has been declared to be a p-form, f to be a scalar, and u to be a vector. Hence we can type in directly

```
(f*u)|_omega;
```

and find

```
d(u _| omega)*f + u _| d omega*f + d f^u _| omega
```

The rule is verified, but Excalc substituted immediately (A.2.51). Anyway, we see that also |_ exactly does what we expect from it.

In Sec. A.2.8, we introduced the frame e_α and the coframe ϑ^α as bases of the tangent and the cotangent space, respectively. In Excalc we use the symbols `e(-a)` and `o(a)`, respectively. In Excalc a coframe can only be specified, provided a metric is given at the same time. This feature of Excalc is not ideal for our purposes. Nevertheless, even if we introduce the metric only in Part C, we have to use it in the Excalc program already here in order to make the programs of Part B executable.

As we saw already in Sec. A.1.12, we can introduce Excalc to the dimension of a manifold via `spacedim 4;` with the `coframe`–statement this can also be done, since we specify thereby the underlying four one-forms of the coframe and, if the coframe is orthonormal, the signature of the metric. For a Minkowski spacetime with time coordinate t and spherical spatial coordinates r, θ, φ, we state

```
coframe   o(t)     =                    d t,
          o(r)     =                    d r,
          o(theta) = r *                d theta,
          o(phi)   = r * sin(theta) * d phi
   with signature (1,-1,-1,-1);
frame e;
```

With `frame e;`, we assigned the identifier e to the name of the frame. In ordinary mathematical language, the coframe statement would read

$$\vartheta^t = dt\,,\quad \vartheta^r = dr\,,\quad \vartheta^\theta = r\,d\theta\,,\quad \vartheta^\phi = r\sin\theta\,d\phi\,,$$
$$g = \vartheta^t\otimes\vartheta^t - \vartheta^r\otimes\vartheta^r - \vartheta^\theta\otimes\vartheta^\theta - \vartheta^\phi\otimes\vartheta^\phi\,. \qquad \text{(A.2.73)}$$

COFRAME

Of course, the frame `e(-a)` and the coframe `o(a)` are inverse to each other, i.e., the command `e(-a) _|o(b);` will yield the Kronecker delta (if you switch `on nero;` then only the components which nonvanishing values will be printed out).

The coframe statement is very fundamental for Excalc. All quantities will be evaluated with respect to this coframe. This yields the anholonomic (or physical) components of an object.

The coframe statement of a corresponding spherically symmetric Riemannian metric with unknown function $\psi(r)$ reads:

```
load_package excalc$
pform psi=0$ fdomain psi=psi(r)$
coframe o(t)     = psi *             d t,
        o(r)     = (1/psi) *         d r,
        o(theta) = r *               d theta,
        o(phi)   = r * sin(theta) *  d phi
   with signature (1,-1,-1,-1)$

displayframe;       % displays the coframe o(a), check for input
frame e$
```

Perhaps we should remind ourselves that $\psi^2 = 1 - 2m/r$ represents the Schwarzschild solution of general relativity.

A.2.12 $^{\otimes}$Closed and exact forms, de Rham cohomology groups

Closed forms are not exact in general. Two closed forms belong to the same cohomology class when they differ by an exact form. Groups of cohomologies are topological invariants.

Let us consider the exterior algebra $\Lambda^*(X) = \bigoplus_{p=0}^{n} \Lambda^p(X)$ together with the exterior derivative defined in (A.2.19).

A p-form ω is called *closed*, if $d\omega = 0$. The space of all closed p-forms

$$Z^p(X) := \{\omega \in \Lambda^p(X) | d\omega = 0\}, \qquad p = 0, \ldots, n, \quad \text{(A.2.74)}$$

forms a (real) vector subspace of $\Lambda^p(X)$. A p-form ω is called *exact*, if a $(p-1)$-form φ exists such that $\omega = d\varphi$. The space of all exact p-forms

$$B^p(X) := \{\omega \in \Lambda^p(X) | \omega = d\varphi\}, \qquad p = 1, \ldots, n, \quad \text{(A.2.75)}$$

is also a (real) vector space, and evidently $B^p(X) \subset Z^p(X)$ (each exact form is closed, since $dd \equiv 0$). One puts $B^0(X) = \emptyset$.

Obviously the exterior derivative defines an equivalence relation in the space of closed forms: two forms $\omega, \omega' \in Z^p(X)$ are said to to be *cohomologically* equivalent if they differ by an exact form, i.e. $(\omega - \omega') \in B^p(X)$. The quotient space

$$H^p(X;\mathbb{R}) := Z^p(X)/B^p(X), \qquad p = 0, \ldots, n, \tag{A.2.76}$$

consists of *cohomology classes* of p-forms. Each $H^p(X;\mathbb{R})$ is a vector space and, moreover, it is an Abelian group with an evident group action. The $H^p(X;\mathbb{R})$ are named as *de Rham cohomology groups.*

Unlike the $\Lambda^p(X)$ which are infinite-dimensional functional space, the de Rham groups, for compact manifolds X, are finite-dimensional. The dimension

$$b^p(X) := \dim H^p(X;\mathbb{R}) \tag{A.2.77}$$

is called the p-th *Betti number* of the manifold X.

Locally, an exterior derivative does not yield a difference between closed and exact forms. This fact is usually formulated as *Poincaré lemma*: Locally, in a given chart (U, ϕ) of X, every closed p-form ω, with $d\omega = 0$, is exact, i.e. a $(p-1)$-form φ exists in $U \subset X$ such that $\omega = d\varphi$. Let us illustrate this by an explicit example. Suppose we have a closed *one*-form ω. In local coordinates,

$$\omega = \omega_i(x)\, dx^i, \tag{A.2.78}$$

$$d\omega = 0 \quad \Longleftrightarrow \quad \partial_i \omega_j(x) = \partial_j \omega_i(x). \tag{A.2.79}$$

dom

Then this form is locally exact, $\omega = d\varphi$, where the zero-form φ is given explicitly in the chart (U, ϕ) by

$$\varphi(x) = \int_0^1 \omega_i(tx)\, x^i\, dt. \tag{A.2.80}$$

lemP

Indeed, let us check directly by differentiation:

$$
\begin{aligned}
d\varphi &= \int_0^1 dt\,\left[(\partial_j \omega_i(tx))\, t\, x^i\, dx^j + \omega_i(tx)\, dx^i\right] \\
&= \int_0^1 dt\,\left[t\, x^j\, \partial_j \omega_i(tx) + \omega_i(tx)\right] dx^i = \int_0^1 dt \frac{d[t\,\omega_i(tx)]}{dt}\, dx^i \\
&= \omega_i(x)\, dx^i = \omega. \qquad \text{(A.2.81)}
\end{aligned}
$$

We used (A.2.79) when moving from the first line to the second one. The explicit construction (A.2.80) is certainly not unique but it is sufficient for demonstrating of how the proof works. One can easily generalize (A.2.80) for the case when ω is a p-form, $p > 1$,

$$
\varphi(x) = \int_0^1 t^{p-1}\, u \lrcorner\, \omega(tx)\, dt, \qquad \text{(A.2.82)}
$$

lemPP

where the vector field u is *locally* defined by $u = x^i \partial_i$; its integral lines evidently form a “star-like” structure with the center at the origin of the local coordinate system.

Globally, i.e. on the whole manifold X, however, not every closed form is exact: one usually states that topological obstructions exist. The importance of de Rham groups is directly related to the fact that they present an example of *topological invariants* of a smooth manifold. Of course, the Betti numbers then also encode information about the topology of X. The zeroth number $b^0(X)$, for instance, simply counts the connected components of any manifold X. This follows from the fact that 0-forms are just functions of X, and hence a closed form φ, with $d\varphi = 0$, is a constant on every connected component. Since there are no exact 0-forms, $B^0(X) = \oslash$, the elements of the group $H^0(X;\mathbb{R})$ are thus N-tuples of constants, with N equal to the number of connected components. Hence $b^0(X) := \dim H^0(X;\mathbb{R}) = N$.

Moreover, recall that in Sec. A.2.9 for any differentiable map $f : X \to Y$ we have described a pull-back map of exterior forms on a manifold Y to the forms on X. Since the pull-back commutes with the exterior derivative, see (A.2.42), we immediately find that any such map determines a map between the relevant cohomology groups:

$$f^* : H^p(Y;\mathbb{R}) \longrightarrow H^p(X;\mathbb{R}). \tag{A.2.83}$$

With the help of this map one can prove a fundamental fact: if X and Y are homotopically equivalent manifolds, their de Rham cohomology groups are isomorphic. As a consequence, their Betti numbers are equal, $b^p(X) = b^p(Y)$. Homotopical equivalence essentially means that the manifolds X and Y can be "continuously deformed into one another". An n-dimensional Euclidean space $\mathbb{E}^n$ is homotopically equivalent to an n-dimensional disk $D^n = \left\{(x^1,\ldots,x^n) \in \mathbb{E}^n \,|\, \sqrt{\sum_{i=1}^n (x^i)^2} \leq 1\right\}$, for example, and both are homotopically equivalent to a point. Another example: a Euclidean plane $\mathbb{E}^2$ with one point (say, origin) removed is homotopically equivalent to a circle S^1. More rigorously, manifolds X and Y are homotopically equivalent, if there are two differentiable maps $f_1 : X \to Y$ and $f_2 : Y \to X$ such that $f_2 \circ f_1 : X \to X$ and $f_1 \circ f_2 : Y \to Y$ are homotopic to identity maps id_X and id_Y, respectively. Two maps are homotopic, if they can be related by a smooth family of maps.

The alternating sum

$$\chi(X) := \sum_{p=0}^{n} (-1)^p \, b^p(X) \tag{A.2.84}$$

`Euler`

is a topological invariant called the *Euler characteristic* of a manifold X. In two dimensions, every orientable closed (compact without a boundary) manifold is diffeomorphic to a sphere with a finite number of handles, $M_h^2 := S^2 +$"h handles", where $h = 0, 1, 2, \ldots$ (for $h = 1$, we find a torus $M_1^2 = \mathbb{T}^2$ from Fig. A.2.3). Euler characteristics of these manifolds is $\chi(M_h^2) = 2 - 2h$. Analogously, for the non-orientable two-dimensional manifolds $N_k^2 := S^2 +$"k cross-caps" (Figs. A.2.4, A.2.5 show $N_1^2 = \mathbb{P}^2$ and $N_2^2 = \mathbb{K}^2$, respectively), the Euler characteristic is equal $\chi(N_k^2) = 2 - k$.

A.3

Integration on a manifold

In this chapter we will describe the *integration of exterior forms on a manifold*. The calculus of differential forms provides us with a powerful technique. This occurs because one theorem, known as the Stokes or the Stokes-Poincaré theorem, replaces a number of different theorems known from 3-dimensional vector calculus. Both types of p-forms, ordinary and twisted ones, can be integrated over p-dimensional submanifolds; and in both cases one needs an additional structure, the orientation, in order to define them. For ordinary forms one needs the *inner* and for twisted forms the *outer* orientation. There are two exceptions: to integrate an ordinary 0-form or a twisted n-form, no orientation is necessary.

A.3.1 Integration of 0-forms and orientability of a manifold

The integral of a 0-form f over a 0-dimensional submanifold (set of points in X) is just a sum of values of f at these points.

Let f be a function on X, i.e. $f \in \Lambda^0(X)$, and let Ξ be a finite collection of points, $\Xi = (p_1, \ldots, p_k)$. We can then define the integral of f over Ξ by

$$\int_{\Xi} f := \sum_{i=1}^{k} f(p_i)\,. \tag{A.3.1}$$ 1

If f is, instead of being an ordinary function, a twisted function, then this definition is not satisfactory. Then the $f(p_i)$'s change their signs together with the change of the orientation of the reference frames at each point p_i. If we fix one of the orientations at, say, the point p_1, then we can try to propagate this orientation by continuity to all other points $p_2, \ldots, p_k$. If this can be done unambiguously, then we say that the manifold is orientable, and we have just chosen an orientation for X. In such a case, the values of the function f, i.e. $f(p_1), \ldots, f(p_k)$, can be taken with respect to any frame with positive orientation, and formula (A.3.1) defines unambiguously the integral $\int_{\Xi} f$ of a twisted 0-form.

A.3.2 Integration of n-forms

The integral of a n-form will be defined over an orientable n-dimensional manifold. In the case of a twisted n-form the orientation is not needed for that purpose.

A support for a p-form φ on X is defined as a set $\mathrm{Supp}(\varphi) := \{p \in X | \varphi(p) \neq 0\}$. Let ω be an n-form on X, i.e. $\omega \in \Lambda^n(X)$. At first, let us consider the case when its support $\mathrm{Supp}(\omega) \subset U$ is contained in *one* coordinate chart (U, ϕ), $\phi = \{x^i\}$. Then in $U \subset X$ we have

$$\omega = f(x)\, dx^1 \wedge \cdots \wedge dx^n\,, \tag{A.3.2}$$ 2

where we denoted the only component of the n-form as $f(x) = \omega_{1\dots n}(x)$, cf. (A.2.10). We can try to define

$$\int\limits_X \omega := \int\limits_{\phi(U)} f(x^1,\dots,x^n)\,dx^1\dots dx^n\,, \tag{A.3.3}$$ 3

where $\phi(U)$ is the image of U in $\mathbb{R}^n$ for the coordinate map ϕ, and on the right-hand side of (A.3.3) we have a usual Riemann integral. The "definition" above is, however, ambiguous if one changes the coordinates in (A.3.2). Without touching U, one can consider, for example, an arbitrary diffeomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$ which will introduce a new local coordinate map $\phi' = A \circ \phi$ with coordinates $\{y^i\}$ in U. Under the change of variables $x^i = x^i(y^j)$, the right-hand side of (A.3.3) transforms into

$$\int\limits_{\phi(U)} f(x)\,dx^1\dots dx^n \;=\; \int\limits_{\phi'(U)} f(y)\,|J(A^{-1})|dy^1\dots dy^n, \tag{A.3.4}$$

where $J(A^{-1}) = \det \partial x^i/\partial y^j$ is the Jacobian determinant of the variable change, cf. (A.2.2). But from (A.3.2) we have $\omega'_{1\dots n}(y) = f(y)\,J(A^{-1})$. Thus the transformed left-hand side of (A.3.3) is equal $\pm$ of the right-hand side, depending on the sign of the Jacobian determinant. A diffeomorphism A which changes the orientation of the coordinate system leads to a change of sign of the integral (A.3.3). One has thus to *fix an orientation* in X in order to have a definite notion of the integral $\int\limits_X \omega$.

If $\mathrm{Supp}(\omega)$ is not contained in the domain of a single coordinate chart, the situation is more complicated. Then, for any atlas $\{(U_\alpha,\phi_\alpha)\}$, one has to use a partition of unity $\{\rho_\alpha\}$ subordinate to the covering $\{U_\alpha\}$ of X. In particular, since $\sum_\alpha \rho_\alpha = 1$, we can represent the n-form as a sum $\omega = \sum_\alpha \omega_\alpha$, where each $\omega_\alpha := \omega\rho_\alpha$ vanishes outside U_α, i.e., $\mathrm{Supp}(\omega_\alpha) \subset U_\alpha$. For every ω_α we can then construct the integral via (A.3.3) and, finally, an integral for an arbitrary n-form is defined by

$$\int\limits_X \omega := \sum_\alpha \int\limits_{\phi_\alpha(U_\alpha)} f_\alpha(x^1_\alpha,\dots,x^n_\alpha)\,dx^1_\alpha\dots dx^n_\alpha\,, \tag{A.3.5}$$ 3a

where $f_\alpha(x^i_\alpha) = \rho_\alpha\,\omega_{1\dots n}(x^i_\alpha)$. The integral of an ordinary n-form can be defined unambiguously only in the case of an orientable manifold. Moreover, one can prove that it is *uniquely* defined over X if the orientation is prescribed. In particular, this definition is invariant under the change of an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ and/or the partition of unity $\{\rho_\alpha\}$.

The situation is quite different, if, instead of an ordinary n-form, we consider a *twisted* n-form. If (A.3.2) holds for a twisted form ω in one coordinate system, then in another one we have

$$\omega' = f' \, dx^{1'} \wedge \cdots \wedge dx^{n'} \, , \qquad \text{(A.3.6)}$$ 4

with

$$f' = \left| \det \left(\frac{\partial x^i}{\partial x^{i'}} \right) \right| f \, . \qquad \text{(A.3.7)}$$ 5

Here the additional sign factor of (A.2.12) has been taken into account. We don't need any orientation in order to fix uniquely the meaning of (A.3.3). If many U_α's cover $\mathrm{Supp}(\omega)$, then we still need the partition of unity, but the consistency in the intersections $U_\alpha \cap U_\beta$ is automatically provided by (A.3.7). Summing up: Any *twisted* n-form ω with compact support can be integrated over an n-dimensional manifold, regardless whether the latter is orientable or not.

A.3.3 Integration of p-forms with $0 < p < n$

The integral of a p-form is defined for a singular p-simplex.

In order to define an integral of a p-form, with $0 < p < n$, some preliminary constructions are needed which introduce suitable p-dimensional domains of X over which one can integrate.

As a first step, one considers p-simplices in $\mathbb{R}^n$. Take a set of $(p+1)$ ordered points $P_0, P_1, \ldots, P_p \in \mathbb{R}^n$ which are independent in the sense that the p vectors $(P_i - P_0)$, with $i = 1, \ldots p$, are linearly independent (recall that $\mathbb{R}^n$ is a linear vector space). One calls a p-dimensional simplex, or simply a *p-simplex*, the closed convex hull spanned by this set of points:

$$\sigma^p := (P_0, P_1, \ldots, P_p). \qquad \text{(A.3.8)}$$

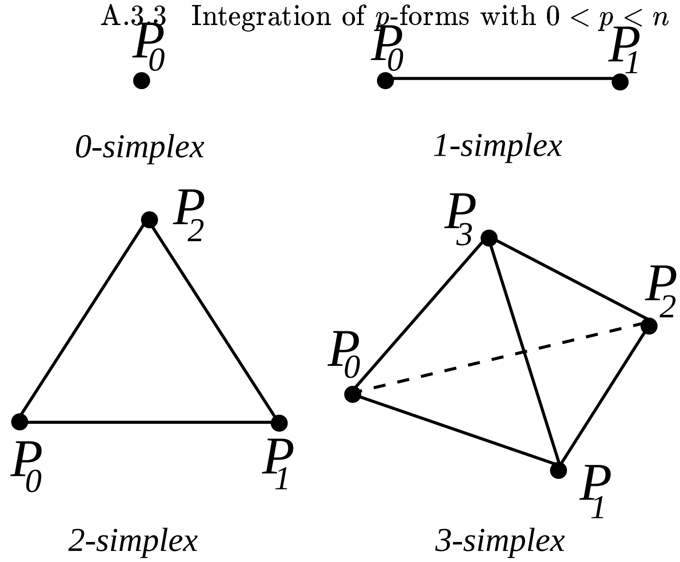

Figure A.3.1: Simplices $\sigma^0, \sigma^1, \sigma^2$, and σ^3.

Geometrically, it is represented by

$$\sigma^p = \left\{ \sum_{i=0}^{p} t^i \, P_i \right\}, \qquad \sum_{i=0}^{p} t^i = 1, \qquad t^i \geq 0, \qquad \text{(A.3.9)}$$ simcoord

where $t^0, \ldots, t^p$ are real numbers.

Accordingly, every 0-simplex is simply one point (P_0); a 1-simplex is a directed line segment (P_0, P_1); a 2-simplex is a closed triangle with ordered vertices (P_0, P_1, P_2); a 3-simplex is a closed tetrahedron (P_0, P_1, P_2, P_3), and so on, see Fig.A.3.1.

Each p-simplex σ^p has a natural $(p-1)$-dimensional boundary which is composed of faces. An i-th *face* $\sigma^{p-1}_{(i)}$, with $0 \geq i \geq p$, of a simplex $(P_0, P_1, \ldots, P_p)$ is defined as a $(p-1)$-simplex $\sigma^{p-1}_{(i)} := (P_0, \ldots, \widehat{P}_i, \ldots, P_p)$ obtained from $(P_0, P_1, \ldots, P_p)$ by removing the vertex P_i (as usual, the hat denotes that an element is omitted from the list). Defined in this way, a face lies opposite to the vertex P_i.

For any p-simplex $\sigma^p = (P_0, P_1, \ldots, P_p)$, with the help of the faces, one can define the formal sum of $(p-1)$-simplices by

$$\partial \sigma^p := \sum_{i=0}^{p} (-1)^i \, \sigma^{p-1}_{(i)}. \qquad \text{(A.3.10)}$$ bound1

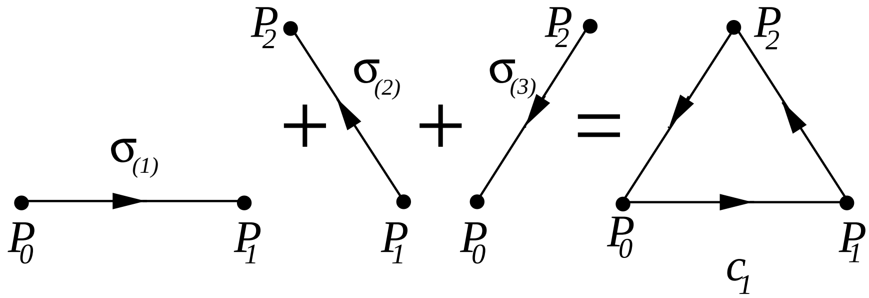

Figure A.3.2: A 1-chain $c_1 = \sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)}$. With $\sigma_{(1)} = (P_0, P_1)$, $\sigma_{(2)} = (P_1, P_2)$, $\sigma_{(3)} = (P_2, P_0)$, the resulting chain is a *boundary* of a 2-simplex: $c_1 = \partial(P_0, P_1, P_2)$. Arrows show the ordering of the vertices.

It is called the *boundary* of σ^p. More explicitly,

$$\partial(P_0, P_1, \ldots, P_p) := \sum_{i=0}^{p} (-1)^i \, (P_0, \ldots, \widehat{P}_i, \ldots, P_p). \quad \text{(A.3.11)}$$ bound2

The boundary of the 2-simplex (P_0, P_1, P_2), for example, reads

$$\partial(P_0, P_1, P_2) = (P_0, P_1) - (P_0, P_2) + (P_1, P_2), \quad \text{(A.3.12)}$$ b2s

see Fig. A.3.2.

The *boundary of a boundary is zero*: for any p-simplex we have

$$\partial\partial(P_0, P_1, \ldots, P_p) = 0. \quad \text{(A.3.13)}$$ bbzero

Let us check this for the 2-simplex (P_0, P_1, P_2). From (A.3.12) and the definition (A.3.11), we find:

$$\begin{aligned}\partial\partial(P_0, P_1, P_2) = {} & [(P_1) - (P_0)] - [(P_2) - (P_0)] \\ & + [(P_2) - (P_1)] = 0. \end{aligned} \quad \text{(A.3.14)}$$

From simplices one is able to construct chains. An arbitrary *p-chain* is a formal sum

$$c_p = \sum_i a_i \, \sigma^p_{(i)}, \quad \text{(A.3.15)}$$ chain

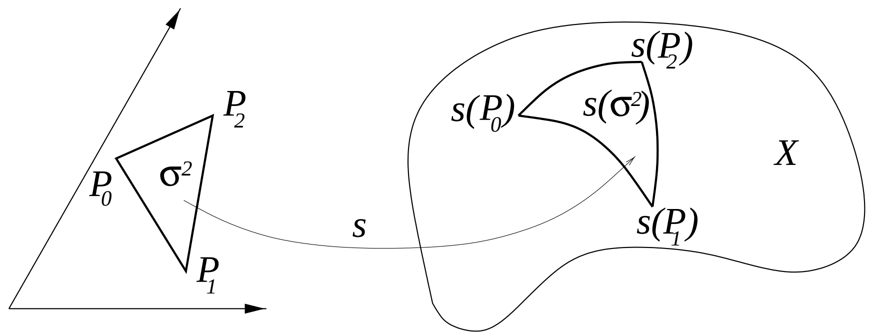

Figure A.3.3: Singular simplices on a smooth manifold X.

where a_i are real coefficients and $\sigma^p_{(i)}$ p-simplices. The boundary of a p-chain is a $(p-1)$-chain defined by

$$\partial c_p = \sum_i a_i \, \partial \sigma^p_{(i)}. \tag{A.3.16}$$ b-chain

In Fig. A.3.2, we demonstrate the construction of a chain from simplices. In this particular case the chain turns out to be a boundary of a 2-simplex.

We are now in a position to define the integral of a p-form on the manifold X. A suitable domain of integration is given by singular simplices in X. Given a p-simplex $\sigma^p \subset \mathbb{R}^p$, a *singular p-simplex* in the manifold X is defined as a differentiable map $s : \sigma^p \to X$. Every point $p \in X$ can evidently be treated as singular 0-simplex, and any smooth curve on X is just a singular 1-simplex, for a 2-simplex see, e.g., Fig. A.3.3, etc.

Consider now a p-form ω on the manifold X. Given a singular p-simplex $s : \sigma^p \to X$, the pull-back s^* maps ω to $\mathbb{R}^p$ and we define the integral of the form over the singular simplex by

$$\int_s \omega := \int_{\sigma^p} f(t^1, \ldots, t^p) \, dt^1 \ldots dt^n, \tag{A.3.17}$$ intpform

where $t^i = (t^1, \ldots, t^p)$ are the standard coordinates in $\mathbb{R}^p$ and $f(t^1, \ldots, t^p) := (s^*\omega)_{1\ldots p}(t^i)$ is the single component of the form $s^*\omega$ on $\sigma^p \subset \mathbb{R}^p$. The right-hand side of (A.3.17) is understood in the usual sense of a Riemann integral.

Like in the case of the integral for n-forms, the questions related to the orientation should be carefully studied separately first for ordinary and then for twisted p-forms.

Let ω now be an *ordinary* p-form on X. No orientation should be specified for X in the definition above. Instead, the preferred (and standard) orientation in $\mathbb{R}^p$ is used. This orientation, by means of the map s, is transported to $s(\sigma)$. In other words, the push-forward s_* maps the standard frame $\left(\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^p}\right)$ to the frame $\left(s_* \frac{\partial}{\partial t^1}, \ldots, s_* \frac{\partial}{\partial t^p}\right)$ tangent to $s(\sigma) \subset X$. This frame determines an inner orientation all over $s(\sigma)$. The value of the integral (A.3.17) is not changed, if we change s and σ^p in such a way (keeping ω untouched) that the orientation induced on $s(\sigma)$ is not changed. Such changes of s to $s' = (s \circ A)$ can be induced by diffeomorphisms $A : \mathbb{R}^p \to \mathbb{R}^p$ which have positive determinant $J(A)$. The value of the integral (A.3.17) does not depend on s; it depends, however, on the choice of *inner* orientation of the simplex σ^p. Therefore it must be assumed that σ^p is inner orientable.

Let us now turn to *twisted* p-forms. In this case, equation (A.3.17) is ambiguous since the choice between $+\omega$ and $-\omega$ depends on the orientation in X, and this fact is not taken care of properly. In order to overcome this ambiguity, we have to determine the *outer* orientation first of the tangent space Ξ_x and subsequently of the whole Ξ. Hereafter we denote a singular p-simplex by $\Xi := s(\sigma^p) \subset X$ and an arbitrary point by $x \in \Xi$. The tangent space of the singular simplex Ξ_x is a subspace of the tangent space X_x at this point. An outer orientation in the tangent space Ξ_x is an orientation in the complementary space X_x/Ξ_x. As usual, it is given by an equivalence class of frames in X_x/Ξ_x which are related by a matrix with positive determinant. In other words, the outer orientation in given by a sequence $(e_{p+1}, \ldots, e_n)$ of linearly independent vectors in X_x which are *transversal* to Ξ. The submanifold is outer orientable, if the outer orientations in the tangent spaces Ξ_x can be chosen continuously on the whole Ξ. Since Ξ is connected, there are only two orientations allowed. If the submanifold is outer oriented, we can require that the sign in front of ω on the right-hand side

(A.3.17) is consistent with the choice of the orientation for the frame

$$\left(s_*\frac{\partial}{\partial t^1}, \ldots, s_*\frac{\partial}{\partial t^p}, e_{p+1}, \ldots, e_n\right) \tag{A.3.18}$$ 8

all over the singular simplex $\Xi = s(\sigma)$. This finally removes, for twisted forms, the ambiguity inherent in (A.3.17). To put it differently, by fixing the outer orientation, the only allowed orientation-reversing coordinate transformations are induced by the orientation-changing diffeomorphisms of $\mathbb{R}^p$, and (A.3.17) is obviously invariant under such transformations.

A good example of the notions introduced so far is the Möbius strip, Fig. A.2.9, considered as a submersed submanifold (with boundary) of $\mathbb{R}^3$. In this case, neither an inner nor an outer orientation can be attached to the Möbius strip in a continuous way. Therefore neither ordinary nor twisted 2-forms, given on $\mathbb{R}^3$, can be integrated over the Möbius strip. However, one can embed the Möbius strip into non-orientable 3-dimensional manifold which is defined as the direct product $X_3 =$"$\mathbb{R}\times$ Möbius strip". In this X_3, the Möbius strip is a two-sided submanifold, and one can thus introduce the outer orientation on it. After fixing the outer orientation, any twisted 2-form on X_3 can be integrated on the Möbius strip.

A.3.4 Stokes's theorem

Stokes's theorem provides for an n-dimensional generalization of the familiar 3-dimensional Gauss and 2-dimensional Stokes theorems.

The Stokes theorem is a far-reaching generalization of the fundamental integration theorem of calculus. Its importance for geometry and physics cannot be overestimated. There are several formulations of Stokes's theorem. Usually, that basic result is presented for an n-dimensional manifold X with a boundary

∂X. Let ω be an $(n-1)$-form on X, then

$$\boxed{\int_X d\omega = \int_{\partial X} \omega}. \qquad \text{(A.3.19)} \quad \texttt{stokesN}$$

This theorem is true for an ordinary form on orientable manifolds as well as for a twisted form on non-orientable manifolds. One can show that, in the former case, a natural inner orientation, and in the latter case, a natural outer orientation is induced on the boundary ∂X. We will not give a rigorous proof here.[1]

Another version of Stokes's theorem, the so-called combinatorial one, is related to the singular homology of a manifold. For any singular p-simplex $s: \sigma^p \to X$ and a $(p-1)$-form ω on the manifold X, the (combinatorial) Stokes theorem states that

$$\int_s d\omega = \int_{\partial s} \omega \,. \qquad \text{(A.3.20)} \quad \texttt{stokes0}$$

Here, in accordance with the definition (A.3.10), the boundary of a singular simplex $s: \sigma^p \to X$ is defined by

$$\partial s(\sigma^p) := \sum_{i=0}^{p} (-1)^i \, s(\sigma^{p-1}_{(i)}). \qquad \text{(A.3.21)} \quad \texttt{bound3}$$

Let us *demonstrate* this theorem *for a 2-simplex.* It is clear that, without loss of generality, one can always choose coordinates (t^1, t^2) in $\mathbb{R}^2 \supset \sigma^2$ in such a way that the vertices of the 2-simplex are the points $P_0 = (0,0)$, $P_1 = (1,0)$, and $P_2 = (0,1)$. The simplex is then called *standard* with the canonical choice of coordinates. The standard 2-simplex is depicted on Fig. A.3.4. Incidentally, the generalization to higher-dimensional simplices in $\mathbb{R}^p$ is straightforward: a standard p-simplex $\sigma^p = (P_0, P_1, \ldots, P_p)$ is defined by the points $P_0 = (0, \ldots, 0)$, $P_1 = (1, 0, \ldots, 0)$, $\ldots$, $P_p = (0, \ldots, 0, 1)$.

Given the parametrization of the standard 2-simplex, cf. (A.3.9),

$$\sigma^2 = \left\{(1 - t^1 - t^2)\, P_0 + t^1\, P_1 + t^2\, P_2\right\}, \qquad 0 \le t^i \le 1, \qquad \text{(A.3.22)} \quad \texttt{stdcoord}$$

its boundary is described by the three 1-simplices (its faces):

$$\begin{aligned} \sigma^1_{(0)} &= \left\{t^1\, P_1 + t^2\, P_2\right\}, & t^1 + t^2 &= 1, \\ \sigma^1_{(1)} &= \left\{t^2\, P_2 + (1 - t^2)\, P_0\right\}, & 0 \le t^2 &\le 1, \\ \sigma^1_{(2)} &= \left\{(1 - t^1)\, P_0 + t^1\, P_1\right\}, & 0 \le t^1 &\le 1. \end{aligned} \qquad \text{(A.3.23)} \quad \texttt{faces}$$

[1] A rigorous proof can be found in Choquet-Bruhat et al.[4], e.g.

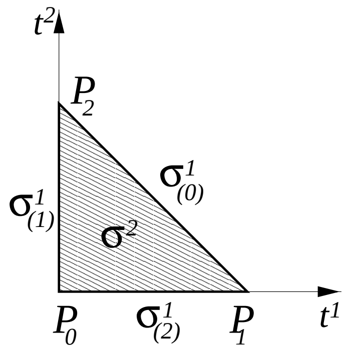

Figure A.3.4: Standard 2-simplex with canonical coordinates on it.

For a 1-form ω on X its pull-back on $\sigma^2 \subset \mathbb{R}^2$ is given by

$$(s^*\omega) = f_1(t^1, t^2)\, dt^1 + f_2(t^1, t^2)\, dt^2, \tag{A.3.24}$$

with the two independent components $f_i(t^1, t^2) = (s^*\omega)_i,\ i = 1, 2$. Accordingly, the exterior derivative reads

$$(s^* d\omega) = \left(\frac{\partial f_2}{\partial t^1} - \frac{\partial f_1}{\partial t^2} \right) dt^1 \wedge dt^2. \tag{A.3.25}$$

Now we have to apply the definition (A.3.17). For the left-hand side of Stokes's theorem we find, using the conventional rules for the multiple integrals,

$$\begin{aligned}
\int\limits_s d\omega = \int\limits_{\sigma^2} s^* d\omega &= \iint\limits_{(P_0,P_1,P_2)} \left(\frac{\partial f_2}{\partial t^1} - \frac{\partial f_1}{\partial t^2} \right) dt^1 dt^2 \\
&= \int\limits_0^1 dt^2 \int\limits_0^{1-t^2} dt^1\, \frac{\partial f_2}{\partial t^1} - \int\limits_0^1 dt^1 \int\limits_0^{1-t^1} dt^2\, \frac{\partial f_1}{\partial t^2} \\
&= \int\limits_0^1 dt^2 \left[f_2(1-t^2, t^2) - f_2(0, t^2) \right] - \int\limits_0^1 dt^1 \left[f_1(t^1, 1-t^1) - f_1(t^1, 0) \right].
\end{aligned} \tag{A.3.26}$$

lhstokes

The right-hand side of Stokes's theorem consists of the three integrals over the faces (A.3.23). Direct calculation of the corresponding line integrals yields:

$$\begin{aligned}
\int\limits_{\sigma^1_{(0)}} s^*\omega &= \int\limits_{P_1}^{P_2} f_i\, dt^i = -\int\limits_0^1 dt^1\, f_1(t^1, 1-t^1) + \int\limits_0^1 dt^2\, f_2(1-t^2, t^2), \\
\int\limits_{\sigma^1_{(1)}} s^*\omega &= \int\limits_{P_2}^{P_0} f_i\, dt^i = \int\limits_0^1 dt^2\, f_2(0, t^2), \\
\int\limits_{\sigma^1_{(2)}} s^*\omega &= \int\limits_{P_0}^{P_1} f_i\, dt^i = \int\limits_0^1 dt^1\, f_1(t^1, 0).
\end{aligned} \tag{A.3.27}$$

rhstokes

Taking into account that

$$\int_{\partial s} \omega = \sum_{i=0}^{p} (-1)^i \int_{\sigma^1_{(i)}} s^* \omega \,, \tag{A.3.28}$$

recall (A.3.21), we compare (A.3.26) and (A.3.27) to verify that (A.3.20) holds true for any 1-form and any singular 2-simplex on X.

A.3.5 ⊗De Rham's theorems

The first theorem of de Rham states that a closed form is exact if and only if all of its periods vanish.

Recall that the de Rham cohomology groups, which were defined in Sec. A.2.12 with the help of the exterior derivative

$$d \,:\, \Lambda^p(X) \longrightarrow \Lambda^{p+1}(X) \tag{A.3.29}$$

dLam

in the algebra of differential forms $\Lambda^*(X)$, "feel" the topology of the manifold X. Likewise, singular simplices can also be used to study the topological properties of X. The relevant mathematical structure is represented by the *singular homology groups.* They are defined as follows: Similarly to a chain as constructed from simplices, see (A.3.15), a *singular p-chain* on a manifold X is defined as a formal sum

$$c_p = \sum_i a_i \, s_i^p, \tag{A.3.30}$$

s-chain

with real coefficients a_i and singular p-simplices s_i^p. In the space $C_p(X)$ of all singular p-chains on X a sum of chains and multiplication by a real constant are defined in an obvious way.

The *boundary* map

$$\partial \,:\, C_p(X) \longrightarrow C_{p-1}(X) \tag{A.3.31}$$

dC

is introduced, in analogy to (A.3.16) and (A.3.21), via defining for every singular p-chain c_p a singular $(p-1)$-chain:

$$\partial c_p = \sum_i a_i \, \partial s_i^p. \tag{A.3.32}$$

bs-chain

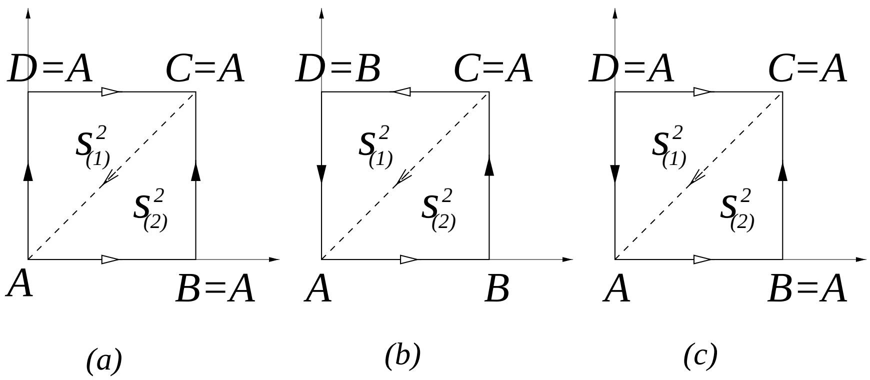

Figure A.3.5: Simplicial decomposition (triangulation) of (a) the torus $\mathbb{T}^2$, (b) the real projective plane $\mathbb{P}^2$, and (c) the Klein bottle $\mathbb{K}^2$.

In complete analogy with the *de Rham complex* $(\Lambda^*(X), d)$, a singular p-simplex z is called a *cycle*, if $\partial z = 0$. The set of all p-cycles,

$$Z_p(X) := \{z \in C_p(X) \mid \partial z = 0\}, \qquad p = 0, \ldots, n, \quad \text{(A.3.33)}$$

is a real vector subspace, $Z_p(X) \subseteq C_p(X)$.

A singular p-chain b is called a *boundary*, if a $(p+1)$-chain b exists such that $b = d\,c$. The space of p-boundaries

$$B_p(X) := \{b \in C_p(X) \mid b = d\,c\}, \qquad p = 1, \ldots, n, \quad \text{(A.3.34)}$$

also forms a (real) vector space and $B_p(X) \subseteq Z_p(X)$, since $\partial\partial \equiv 0$. One puts $B_n(X) = \oslash$.

Finally, the singular homology groups are defined as the quotient spaces

$$H_p(X;\mathbb{R}) := Z_p(X)/B_p(X), \qquad p = 0, \ldots, n. \qquad \text{(A.3.35)}$$

As an instructive example, let us briefly analyze the homological structure of the simplest compact two-dimensional manifolds: The sphere S^2, the torus $\mathbb{T}^2$ (these two are orientable), the real projective plain $\mathbb{P}^2$, and the Klein bottle $\mathbb{K}^2$ (these are non-orientable). The three last manifolds are seen in Fig. A.2.3, Fig. A.2.4, and Fig. A.2.5, respectively. A standard approach to the calculation of homologies for a manifold X is to *triangulate* it, that is to subdivide X into simplices in such a way that the resulting totality of simplices (called a simplicial complex) contains, together with each simplex, also all of its faces; every two simplices either do not have common points or they intersect over a common face of lower dimension. The triangulation of a sphere obviously reduces just to a collection of four 2-simplices which form the boundary of a 3-simplex, that is the surface of a tetrahedron (see Fig. A.3.1). The triangulations of the torus, the projective plane, and the Klein bottle are depicted in Fig. A.3.5.

1) S^2 has as the only 2-cycle the manifold itself, $z^2 = S^2$. Direct inspection shows that there are no non-trivial 1-cycles (they all are boundaries of 2-dimensional chains). Finally, each vertex of the tetrahedron is trivially a 0-cycle, and they are all homological to each other because of the connectedness of S^2. These facts are summarized by displaying the homology groups explicitly:

$$\begin{aligned} H_2(S^2;\mathbb{R}) &= \mathbb{R}, \\ H_1(S^2;\mathbb{R}) &= 0, \\ H_0(S^2;\mathbb{R}) &= \mathbb{R}. \end{aligned} \tag{A.3.36}$$

S-ex

2) $\mathbb{T}^2$ is "composed" of two 2-simplices, $\mathbb{T}^2 = S^2_{(1)} + S^2_{(2)}$, namely $S^2_{(1)} = (A, C, D)$ and $S^2_{(2)} = (A, B, C)$ with the corresponding identifications (gluing) of sides and points as shown in Fig. A.3.5(a). The direct calculation of the boundaries yields $\partial S^1_{(2)} = (A, B) + (B, C) - (A, C)$ and $\partial S^2_{(2)} = (C, D) - (A, D) + (A, C)$. Taking the identifications into account, we then find $\partial\mathbb{T}^2 = 0$, hence the torus itself is a 2-cycle. There are no other non-trivial 2-cycles. As for the 1-cycles, we find *two*: $z^1_{(1)} = (A, B)|_{B=A}$ and $z^1_{(2)} = (B, C)|_{C=B=A}$ (end points are identified). Geometrically, these cycles are just closed curves, one of which goes along and another across the handle. There are no other independent 1-cycles [$z^1_{(3)} = (C, A)|_{C=A}$, for example, is homological to the sum of $z^1_{(1)}$ and $z^1_{(2)}$]. Thus we have verified that the the 1st homology group is two-dimensional. For 0-cycles the situation is exactly the same as for the sphere. In summary, we have for the torus:

$$\begin{aligned} H_2(\mathbb{T}^2;\mathbb{R}) &= \mathbb{R}, \\ H_1(\mathbb{T}^2;\mathbb{R}) &= \mathbb{R}^2, \\ H_0(\mathbb{T}^2;\mathbb{R}) &= \mathbb{R}. \end{aligned} \tag{A.3.37}$$

T-ex

3) $\mathbb{P}^2 = S^2_{(1)} + S^2_{(2)}$, where $S^2_{(1)} = (A, C, D)$ and $S^2_{(2)} = (A, B, C)$ with sides and points identified as shown in Fig. A.3.5b. Repeating the calculation for the torus, we find $\partial\mathbb{P}^2 = c^1_{(1)} + c^1_{(2)}$, where the 1-chains are $c^1_{(1)} = (A, B)$ and $c^1_{(2)} = (B, C)$. Thus, the projective plain itself is not a 2-cycle. Since there are no other homologically inequivalent 2-cycles, we conclude that the 2nd homology group is trivial. Moreover, we immediately verify $\partial c^1_{(1)} = (B) - (A)$ and $\partial c^1_{(2)} = (A) - (B)$ thus proving that $z^1 = c^1_{(1)} + c^1_{(2)}$ is a 1-cycle. Moreover, it is a boundary because of $z^1 = \partial\mathbb{P}^2$. No other 1-cycles exists in $\mathbb{P}^2$. Thus we conclude that the 1st homology group is also trivial. Because of connectedness, the final list reads:

$$\begin{aligned} H_2(\mathbb{P}^2;\mathbb{R}) &= 0, \\ H_1(\mathbb{P}^2;\mathbb{R}) &= 0, \\ H_0(\mathbb{P}^2;\mathbb{R}) &= \mathbb{R}. \end{aligned} \tag{A.3.38}$$

P-ex

4) $\mathbb{K}^2 = S^2_{(1)} + S^2_{(2)}$, where $S^2_{(1)} = (A, C, D)$ and $S^2_{(2)} = (A, B, C)$ with sides and points glued as shown in Fig. A.3.5c. By an analogous calculation, we find $\partial\mathbb{K}^2 = 2(B, C)$. There are no non-trivial 2-cycles on the Klein bottle. Like for the torus, there are two independent 1-cycles, $z^1_{(1)} = (A,B)|_{B=A}$ and $z^1_{(2)} = (B,C)|_{C=B=A}$. However, the second one is a boundary $z^1_{(2)} = \partial\mathbb{K}^2$. Hence $z^1_{(1)}$ generates the only non-trivial homology class for Klein bottle. Thus finally, the homology groups are:

$$\begin{aligned} H_2(\mathbb{K}^2;\mathbb{R}) &= 0, \\ H_1(\mathbb{K}^2;\mathbb{R}) &= \mathbb{R}, \\ H_0(\mathbb{K}^2;\mathbb{R}) &= \mathbb{R}. \end{aligned} \tag{A.3.39}$$ K-ex

Similarly to the de Rham cohomology groups $H^p(X;\mathbb{R})$, see Sec. A.2.12, the singular homology groups $H_p(X;\mathbb{R})$ are *topological invariants* of a manifold. In particular, they do not change under a 'smooth deformation' of a manifold, i.e., they are homotopically invariant. Cohomologies and homologies are deeply related. In order to demonstrate this (although without rigorous proofs), we need the central notion of a period. For any *closed* p-form $\omega \in Z^p(X)$ and each *p-cycle* $z \in Z_p(X)$, a *period* of the form ω is the number

$$\mathrm{per}_z(\omega) := \int_z \omega. \tag{A.3.40}$$ period

This real number is not merely a function of ω and z: it rather depends on the whole cohomology class of the form, $[\omega] \in H^p(X;\mathbb{R})$, and on the whole homology class of the cycle $[z] \in H_p(X;\mathbb{R})$. Stokes's theorem underlies the proof: for any cohomologically equivalent p-form, $\omega + d\varphi$, and for any homologically equivalent p-cycle, $z + \partial c$, we find

$$\begin{aligned} \mathrm{per}_z(\omega + d\varphi) &= \int_z (\omega + d\varphi) = \int_z \omega + \int_{\partial z} \varphi = \int_z \omega = \mathrm{per}_z(\omega), \\ \mathrm{per}_{z+\partial c}(\omega) &= \int_{z+\partial c} \omega = \int_z \omega + \int_z d\omega = \int_z \omega = \mathrm{per}_z(\omega), \end{aligned} \tag{A.3.41}$$

since $\partial z = d\omega = 0$. Therefore, in a strict sense, one has to write a period as $\mathrm{per}_{[z]}([\omega])$. We recall the definition of a form as a linear map from a vector space V to the reals, see Sec. A.1.1. Accordingly, one can treat the period (A.3.40) as a 1-form on the space of cohomologies, with $V = H^p(X;\mathbb{R})$, i.e., as an element of the dual space

$$\mathrm{per}_{[z]}([\omega]) \in H_p(X;\mathbb{R})^* \qquad \text{for all } [\omega] \in H^p(X;\mathbb{R}). \tag{A.3.42}$$

DRmap

The linear map $\mathrm{DR} : H^p(X;\mathbb{R}) \to H_p(X;\mathbb{R})^*$, defined via the equations (A.3.42) and (A.3.40) as $\mathrm{DR}([\omega])([z]) := \mathrm{per}_{[z]}([\omega])$, is called *de Rham map*. A fundamental *theorem of de Rham* states that this map is an isomorphism. Sometimes, the proof of de Rham theorem is subdivided into the two separate propositions known as the *first* and *second* de Rham theorems. The *first de Rham theorem* reads: *A closed form is exact if and only if all of its periods vanish*:

$$\left\{\begin{matrix} \omega \in Z^p(X) \\ \\ \Downarrow \\ \\ \omega \in B^p(X) \end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} \mathrm{per}_z(\omega) = 0 \\ \\ \text{for all} \\ \\ z \in Z_p(X) \end{matrix}\right\}. \tag{A.3.43}$$

firstDR

In simple terms, the first theorem tells that $\mathrm{DR}([\omega]) = 0 \Leftrightarrow [\omega] = 0$. A 1-form on a vector space V is determined by its components which give the values of that form with respect to a basis of V. Suppose we have chosen a basis $[z_i]$ of the p-th homology group $H_p(X;\mathbb{R})$, i.e., a complete set of homologically inequivalent singular p-cycles z_i. [For a compact manifolds this set is finite.] Denote as $\Theta^i \in V^* = H_p(X;\mathbb{R})^*$ the dual basis to $[z_i]$. Each 1-form on $V = H_p(X;\mathbb{R})$ is then an element $a_i\,\Theta^i$ specified by a set of real numbers $\{a_i\}$ with i running over the whole range of the basis z_i. The *second de Rham theorem* states that the de Rham map *is invertible*, that is, for every set of real numbers $\{a_i\}$ there exists a closed p-form ω on X such that

$$\mathrm{DR}([\omega]) = a_i\,\Theta^i, \qquad \text{i.e.} \qquad [\omega] = \mathrm{DR}^{-1}(a_i\,\Theta^i). \tag{A.3.44}$$

secondDR

In combination with the second theorem, the first de Rham theorem clearly guarantees that the de Rham map is one-to-one: Suppose that for a given set $\{a_i\}$ one can find two 1-forms ω and ω' which both satisfy (A.3.44). Then we get $\mathrm{DR}([\omega - \omega']) = 0$, and (A.3.43) yields $[\omega] = [\omega']$, i.e. ω and ω' differ by an exact form.

Incidentally, our earlier study of the homological structure of the two-dimensional manifolds $S^2, \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2$ gave explicit constructions of the bases $[z_i]$ of the homology groups. One can show that for an arbitrary *compact* manifold X both, the cohomology and homology groups, are finite-dimensional vector spaces. Then the de Rham map establishes the canonical isomorphism

$$H_p(X;\mathbb{R}) \overset{\mathrm{DR}}{\simeq} H^p(X;\mathbb{R}), \qquad p = 0, \ldots n. \tag{A.3.45}$$ DRiso

In particular, $\dim H^p(X;\mathbb{R}) = \dim H_p(X;\mathbb{R})$. Then one can, for example, calculate the Euler characteristics (A.2.84) easily. Returning again to the 2-dimensional examples, we find: $\chi(S^2) = 1 - 0 + 1 = 2$, see (A.3.36); $\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0$, see (A.3.37); $\chi(\mathbb{P}^2) = 0 - 0 + 1 = 1$, see (A.3.38); and $\chi(\mathbb{K}^2) = 0 - 1 + 1 = 0$, see (A.3.39).

Problem:

Show that properties 1)-4) lead uniquely to the formula (A.2.16), i.e., they provide also a definition of the exterior derivative.

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Part B

Axioms of classical electrodynamics

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In this chapter we want to put phenomenological classical electrodynamics into such a form that the underlying physical facts are clearly visible. We will recognize that the conservation of *electric charge* and of *magnetic flux* are two main experimentally well-founded axioms of electrodynamics. For their formulation we will take exterior calculus, because it is the appropriate mathematical framework for handling fields the integrals of which – here charge and flux – possess an invariant meaning.

The densities of the electric charge and the electric current are assumed to be phenomenologically specified. These quantities will not be resolved any further and will be considered as fundamental for classical electrodynamics.

B.1

Electric charge conservation

...it is now discovered and demonstrated, both here and in Europe, that the Electrical Fire is a real Element, or Species of Matter, not created *by the Friction, but* collected *only.*

Benjamin Franklin (1747)[1]

B.1.1 Counting charges. Absolute and physical dimension

Progress in semiconductor technology has enabled the fabrication of structures so small that they can contain just one mobile electron. By varying controllably the number of electrons in these 'artificial atoms' and measuring the energy required to add suc-

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[1] See Heilbron [15] page 330. On the same page Heilbron states: "Although Franklin did not 'discover' conservation, he was unquestionably the first to exploit the concept fruitfully. Its full utility appeared in his classic analysis of the condenser." Note that the discovery of charge conservation preceeded the discovery of the Coulomb law (1785) by more than 40 years. This historical sequence is reflected in our axiomatics. However, our reason is not a historical but a conceptual one: charge conservation should come first and the Maxwell equations be formulated such as to be compatible with that law.

cessive electrons, one can conduct atom physics experiments in a regime that is inaccessible to experiments on real atoms.

R.C. Ashoori (1996)[2]

Phenomenologically speaking, electromagnetism has two types of sources: The electric charge density ρ and the electric current density $\vec{j}$. The electric current density can be understood, with respect to some reference system, as moving electric charge density.

In this section we will give a *heuristic* discussion of charge conservation which will be used, in the next section, as a motivation to formulate a first axiom for electrodynamics.

Imagine a 3-dimensional simply connected region Ω_3, which is enclosed by the 2-dimensional boundary $\partial\Omega_3$, see Fig.B.1.1. In the region Ω_3, there are elementary particles with charge $\pm e$ (e = elementary electric charge) and quarks with charges $\pm\frac{1}{3}e$ and $\pm\frac{2}{3}e$, respectively, which move all with some velocity. It is assumed in electrodynamics that we can attribute to the 3-region Ω_3, at any time in a certain reference system, a well-defined net charge Q with *absolute dimension* charge q (in SI-units Coulomb, abbreviated C):

$$Q = \int_{\Omega_3} \rho \,, \qquad [Q] = q \,, \quad [\rho] = q \,. \tag{B.1.1}$$ charge

Here $[Q]$ should be read as "dimension of Q" and, analogously, $[\rho]$ as "dimension of ρ."

The integrand ρ, also with absolute dimension of charge q, is called the charge-density 3-form. It assigns to the volume 3-form in an arbitrary coordinate system $(a, b, \ldots = 1, 2, 3)$

$$dx^a \wedge dx^b \wedge dx^c \,, \tag{B.1.2}$$ vol

a scalar-valued charge

$$\rho = \frac{1}{3!}\,\rho_{abc}\,dx^a \wedge dx^b \wedge dx^c \,, \qquad \rho_{abc} = \rho_{[abc]} \,, \tag{B.1.3}$$ chargedens

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[2] See his article [1]. For a review on the the counting of single electrons, see Devoret and Grabert [5].

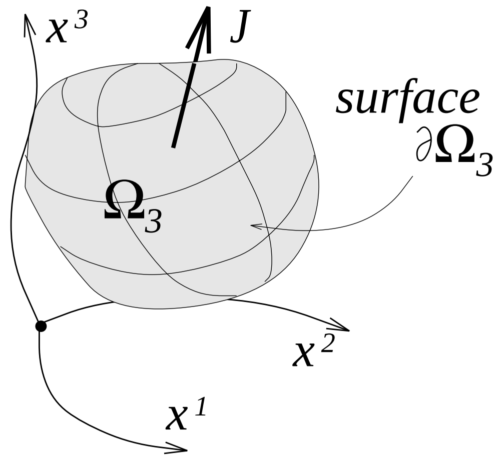

Figure B.1.1: Charge conservation in 3-dimensional space.

which, as scalar, can be added up in any coordinate system to yield Q. Thus, as already noted, even the charge *density* ρ carries the same absolute dimension as the net charge Q.

In spatial spherical coordinates (r, θ, ϕ), for instance, the coordinates carry different dimensions: The r-coordinate has the dimension of a length, whereas θ and ϕ are dimensionless. Therefore we introduce an arbitrary 3-dimensional local coframe ϑ^μ, with $\mu, \nu, \ldots = 1, 2, 3$, and its dual frame e_ν, with $e_\nu \rfloor \vartheta^\mu = \delta^\mu_\nu$. We assign to each of the three 1-forms ϑ^μ the *dimension of a length* ℓ and to the corresponding vectors e_ν the dimension of ℓ^{-1}:

$$[\vartheta^\mu] = [(\vartheta^{\hat{1}}, \vartheta^{\hat{2}}, \vartheta^{\hat{3}})] = (\ell, \ell, \ell)\,, \qquad \text{(B.1.4)}$$

coframedim

$$[e_\nu] = [(e_{\hat{1}}, e_{\hat{2}}, e_{\hat{3}})] = (\ell^{-1}, \ell^{-1}, \ell^{-1})\,. \qquad \text{(B.1.5)}$$

Length is here understood as a *segment*, that is, as a 1-dimensional extension in affine geometry, not, however, as a distance in the sense of Euclidean geometry.

Now we can decompose the charge density 3-form with respect to the coframe ϑ^μ,

$$\rho = \frac{1}{3!}\,\rho_{\mu\nu\lambda}\,\vartheta^\mu \wedge \vartheta^\nu \wedge \vartheta^\lambda\,, \quad \text{phys.dim.of } \rho := [\rho_{\mu\nu\lambda}] = q\,\ell^{-3}\,. \tag{B.1.6}$$

chargedim

The dimension of the anholonomic components $\rho_{\mu\nu\lambda}$ of the charge density ρ is called the *physical dimension* of ρ. In the *hypothesis of locality*[3] it is assumed that the measuring apparatuses in the coframe ϑ^μ, even if the latter is accelerated, measure the anholonomic components of a physical quantity, such as the components $\rho_{\mu\nu\lambda}$, exactly as in a momentarily comoving inertial frame of reference. In the special case of the measurment of time, Einstein spoke about the clock hypothesis.

If we assume, as suggested by experience, that the electric charge Q has *no intrinsic screw-sense*, then the sign of Q does not depend on the orientation in space. Accordingly, the charge density ρ is represented by a *twisted* 3-form; for the definition of twisted quantities, see the end of Sec. A.1.3.

Provided the coordinates x^a are given in Ω_3, we can compute dx^a and the volume 3-form (B.1.2). There is no need to use a metric nor a connection, the properties of a 'bare' differential manifold (continuum) are sufficient for the definition of (B.1.1). This can also be recognized as follows:

The net charge Q in (B.1.1) can be determined by counting the charged elementary particles inside $\partial\Omega_3$ and adding up their elementary electric charges. Nowadays one catches single electrons in traps. Thus the counting of electrons is an experimentally feasible procedure, not only a thought experiment devised by a theoretician. This consideration shows that it is not necessary to use a distance concept nor a length standard in Ω_3 for the determination of Q. Only 'counting procedures' are required and a way to delimit an arbitrary finite volume Ω_3 of

[3]The formulation of Mashhoon [25] reads: "...the hypothesis of locality — i.e., the presumed equivalence of an accelerated observer with a momentarily comoving inertial observer — underlies the standard relativistic formalism by relating the measurements of an accelerated observer to those of an inertial observer."

3-dimensional space by a boundary $\partial\Omega_3$ and to know what is inside $\partial\Omega_3$ and what outside. Accordingly, ρ is the *prototype* of a charge density with absolute dimension $[\rho] = q$ and physical dimension $[\rho_{abc}] = q\ell^{-3}$. It becomes the conventional charge density, that is charge per *scaled* unit volume, if a unit of distance (m in SI-units) is introduced additionally. Then, in SI-units, we have $[\rho_{\mu\nu\lambda}] = C\, m^{-3}$.

Out of the region Ω_3, crossing its bounding surface $\partial\Omega_3$, there will, in general, flow a net electric current, see Fig.B.1.1,

$$\mathcal{J} = \oint_{\partial\Omega_3} j\,, \tag{B.1.7}$$

cur

with absolute dimension $q\,t^{-1}$ ($t =$ time), which must not depend on the orientation of space either. The integrand j, the twisted charge current-density 2-form with the same absolute dimension $q\,t^{-1}$, assigns to the area element 2-form

$$dx^a \wedge dx^b \tag{B.1.8}$$

area

a scalar-valued charge current

$$j = \frac{1}{2!}\, j_{ab}\, dx^a \wedge dx^b, \qquad j_{ab} = j_{[ab]}\,. \tag{B.1.9}$$

curdens

The postulate of electric charge conservation requires

$$\frac{dQ}{dt} = -\mathcal{J}\,, \tag{B.1.10}$$

cons1

provided the area 2-form $dx^a \wedge dx^b$ is directed in such a way that the outflow is counted positively in (B.1.7). The time variable t, provisionally introduced here, does not need to possess a scale or a unit. It can be called a 'smooth causal time' in the sense of parameterizing a future directed curves in the spacetime manifold with t as a monotone increasing and sufficiently smooth variable. Substitution of (B.1.1) and (B.1.7) into (B.1.10) yields an integral form of charge conservation:

$$\frac{d}{dt}\int_{\Omega_3} \rho + \oint_{\partial\Omega_3} j = 0\,. \tag{B.1.11}$$

cons2

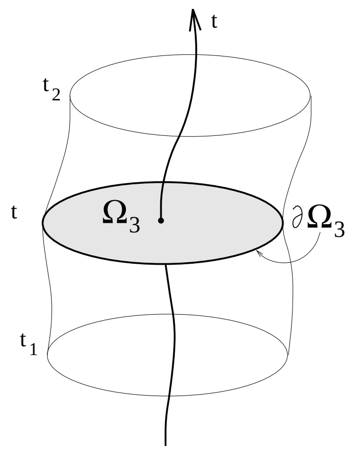

Figure B.1.2: Charge conservation in 4-dimensional spacetime.

By applying the 3-dimensional Stokes theorem, the differential version turns out to be

$$\frac{\partial\rho}{\partial t} + d\,j = 0\,. \tag{B.1.12}$$
cons3

Let us put (B.1.11) into a 4-dimensional form. For this purpose we integrate (B.1.11) over a certain time interval from t_1 to t_2, see Fig.B.1.2. Note that this figure depicts the same physical situation as in Fig.B.1.1:

$$\begin{aligned} &\int\limits_{t_1}^{t_2} dt\,\frac{d}{dt}\int\limits_{\Omega_3}\rho + \int\limits_{t_1}^{t_2} dt\wedge\oint\limits_{\partial\Omega_3} j \\ &= \int\limits_{t=t_2,\,\Omega_3}\rho - \int\limits_{t=t_1,\,\Omega_3}\rho + \int\limits_{[t_1,t_2]\times\partial\Omega_3} dt\wedge j = 0\,. \end{aligned} \tag{B.1.13}$$
cons4

Obviously we are integrating over a 3-dimensional boundary of a compact piece of the 4-dimensional spacetime. If we introduce in 4 dimensions the twisted 3-form

$$J := -j\wedge dt + \rho\,, \tag{B.1.14}$$
4cur

then the integral can be written as a 4-dimensional boundary integral,

$$\oint_{\partial\Omega_4} J = 0\,, \qquad \text{(B.1.15)}$$ cons5

where $\Omega_4 = [t_1, t_2] \times \Omega_3$. The twisted 3-form J of the electric current with dimension q plays the central role as source of the electromagnetic field.

B.1.2 Spacetime and the first axiom

Motivated by the integral form of charge conservation (B.1.15), we can now turn to an axiomatic approach of electrodynamics. First we will formulate a set of minimal assumption that we shall need for defining an appropriate spacetime manifold.

Let spacetime be given as a 4-dimensional *connected, Hausdorff, orientable, and paracompact* differentiable manifold X_4. This manifold is 'bare', that is, it carries neither a connection nor a metric so far. We assume, however, the conventional continuity and differentiability requirements of physics. To recall, a topological space X is Hausdorff when for any two points $p_1 \neq p_2 \in X$ one can find open sets $p_1 \in U_1 \subset X$, $p_2 \in U_2 \subset X$, such that $U_1 \cap U_2 = \oslash$. An X is connected when any two points can be connected by a continuous curve. Finally, a connected Hausdorff manifold is paracompact when X can be covered by a countable number of coordinate charts. The (smooth) coordinates in arbitrary charts will be called x^i, with $i, j, k \ldots = 0, 1, 2, 3$. The vector basis (frame) of the tangent space will be called e_α, the 1-form basis (coframe) of the cotangent space ϑ^α, with (anholonomic) indices $\alpha, \beta, \gamma \cdots = 0, 1, 2, 3$. On the X_4 we can define twisted and ordinary untwisted tensor-valued differential forms.

In order to avoid possible violations of causality, we will, as usual, consider only *non-compact* spacetime manifolds X_4.

The X_4 with the described topological properties is known to possess a $(1+3)$-foliation, see Fig.B.1.3, i.e., there exists a set of

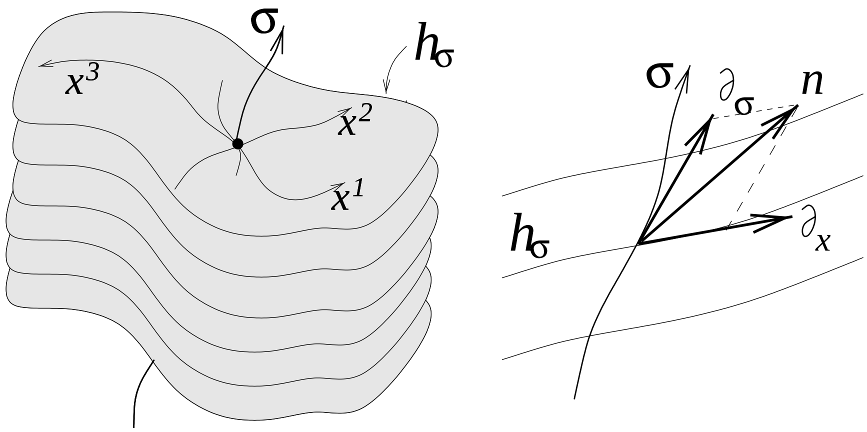

Figure B.1.3: Spacetime and its (1+3)-foliation.

non-intersecting 3-dimensional hypersurfaces h_σ that can be parameterized by a monotone increasing (would-be time) variable σ. Although at this stage we do not introduce any metric on X_4, it is well known that the existence of a $(1+3)$-foliation is closely related to the existence of pseudo-Riemannian structures.

Among the vector fields *transverse* to the foliation, we choose a vector field n, normalized by the condition

$$n \rfloor d\sigma = \mathcal{L}_n \sigma = 1\,. \qquad \text{(B.1.16)}$$ norm

Physically, the folia h_σ of constant σ represent the simple model of a "3-space", whereas the function σ serves as a "time" variable. The vector field n is usually interpreted as a congruence of observer's worldlines. In Sec. E.4.1, this here rather formal mathematical construction becomes a full-fledged physical tool when the metric is introduced.

Now we are in a position to formulate our first axiom. We require the existence of the *twisted charge current 3-form* J which, if integrated over an arbitrary *closed* 3-dimensional submanifold $C_3 \subset X_4$, obeys

$$\oint_{C_3} J = 0\,, \qquad \partial C_3 = 0 \qquad \text{(first axiom)}\,. \qquad \text{(B.1.17)}$$ axiom1

We recall, a manifold is closed if it is compact and has no boundary. In particular, the 3-dimensional boundary $C_3 = \partial\Omega_4$ of an arbitrary 4-dimensional region Ω_4 is a closed manifold. However, in general, not every closed 3-manifold is a 3-boundary of some spacetime region.

This is the first axiom of electrodynamics. It has a firm phenomenological basis.

B.1.3 Electromagnetic excitation

Since (B.1.17) holds for an arbitrary 3-dimensional boundary C_3, we can choose $C_3 = \partial\Omega_4$. Then, by Stokes' theorem, one finds

$$\int_{\Omega_4} d\,J = 0\,. \tag{B.1.18}$$ dj

Since Ω_4 can be chosen arbitrarily, the electric current turns out to be a closed form:

$$d\,J = 0\,. \tag{B.1.19}$$ closed

This is, in 4 dimensions, the differential version of charge conservation.

After having proved that J is a closed 3-form, we now recognize (B.1.17) as the statement that all periods of the current J vanish. Then, by de Rham's first theorem, the current is also an exact form:

$$J = d\,H\,. \tag{B.1.20}$$ curexact

This is the inhomogeneous Maxwell equation. The twisted electromagnetic excitation 2-form H has the absolute dimension of charge q, i.e., $[H] = q$.

The excitation H in this set-up appears as a potential of the electric current. It is determined only up to a closed 2-form

$$H \longrightarrow H + \psi\,, \qquad d\,\psi = 0\,. \tag{B.1.21}$$ excit

In Sec.B.3.4, however, we will discuss how a *unique* excitation field H is selected from the multitude of H's occurring in (B.1.21) by a very weak assumption: The field strength F, to be defined below via the second axiom in (B.2.6), if it vanishes, implies a vanishing excitation H. In this context we will recognize that H can be *measured* by means of an ideal electric conductor and a superconductor of type II, respectively.

Charge conservation, as formulated in the first axiom, is experimentally verified in all microscopic experiments, in particular in those of high-energy elementary particle physics. Therefore the excitation H represents a *microscopic* field as well. Charge conservation is *not* only valid as a macroscopic average. Simlarily, the excitation is not only a quantity which shows up in macrophysics, it rather is a microscopic field, too, analogous to the electromagnetic field strength F to be introduced below.

B.1.4 Time-space decomposition of the inhomogeneous Maxwell equation

"Time is nature's way of keeping everything from happening at once."

Anonymous

Given the spacetime foliation, we can decompose any exterior form in 'time' and 'space' pieces. With respect to the fixed vector field n, normalized by (B.1.16), we define, for a p-form Ψ, the part *longitudinal* to the vector n by

$$^{\perp}\Psi := d\sigma \wedge \Psi_{\perp}, \qquad \Psi_{\perp} := n \rfloor \Psi, \qquad \text{(B.1.22)}$$ longi

and the part *transversal* to the vector n by

$$\underline{\Psi} := (1 - {}^{\perp})\Psi = n \rfloor (d\sigma \wedge \Psi)\,, \qquad n \rfloor \underline{\Psi} \equiv 0. \quad \text{(B.1.23)}$$ trans

Thus the projection operators "$^{\perp}$" and "_" form a complete set. Furthermore, every 0-form is transversal wheras every n-form is longitudinal.

In order to apply this decomposition to field theory, the following rules for the exterior multiplication can be derived from (B.1.22) and (B.1.23),

$$^{\perp}(\Psi \wedge \Phi) = {}^{\perp}\Psi \wedge \underline{\Phi} + \underline{\Psi} \wedge {}^{\perp}\Phi = d\sigma \wedge \left(\Psi_{\perp} \wedge \underline{\Phi} + (-1)^{p}\underline{\Psi} \wedge \Phi_{\perp}\right), \tag{B.1.24}$$
rule1

$$\underline{\Psi \wedge \Phi} = \underline{\Psi} \wedge \underline{\Phi}, \tag{B.1.25}$$
rule2

if Ψ is a p-form. According to (B.1.23), we introduce for the 3-dimensional exterior derivative the notation $\underline{d} := n \rfloor (d\sigma \wedge d)$. Then the exterior derivative of a p-form decomposes as follows:

$$^{\perp}(d\Psi) = d\sigma \wedge (\mathcal{L}_n\underline{\Psi} - \underline{d}\Psi_{\perp}), \qquad \underline{d\Psi} = \underline{d}\,\underline{\Psi}. \tag{B.1.26}$$
decompex

According to (A.2.51), the *Lie derivative* of a p-form along a vector field ξ can be written as

$$\mathcal{L}_{\xi}\Psi := \xi \rfloor d\Psi + d(\xi \rfloor \Psi). \tag{B.1.27}$$
rule4

Notice that the Lie derivative along the foliation vector field n commutes with the projection operators as well as with the exterior derivative, i.e.

$$\mathcal{L}_n {}^{\perp}\Psi = {}^{\perp}(\mathcal{L}_n\Psi), \quad \mathcal{L}_n\underline{\Psi} = \underline{\mathcal{L}_n\Psi}, \quad \mathcal{L}_n d\Psi = d(n \rfloor d\Psi) = d(\mathcal{L}_n\Psi). \tag{B.1.28}$$
commute

These rules also imply that

$$\mathcal{L}_n\,\underline{d}\,\underline{\Psi} = \underline{d}\,\mathcal{L}_n\underline{\Psi}. \tag{B.1.29}$$
rule3

The Lie derivative of the transversal piece $\underline{\Psi}$ of a form with respect to the vector n will be abbreviated by a dot,

$$\dot{\underline{\Psi}} := \mathcal{L}_n\underline{\Psi}, \tag{B.1.30}$$
dot

since this will turn out to be the time derivative of the corresponding quantity.

In order to make this decomposition formalism more transparent, it is instructive to consider natural (co)frames on a 3-dimensional hypersurface h_σ of constant σ. Let $x^a = (x^1, x^2, x^3)$, with $a = 1, 2, 3$, be local coordinates on h_σ. Note that the differentials dx^a are *not* transversal, in general. Indeed, in terms of the local spacetime coordinates (σ, x^a), the normalization (B.1.16) allows for a vector field of the structure $n = \partial_\sigma + n^a\,\partial_a$ with some (in general, nonvanishing) functions n^a, where $a = 1, 2, 3$. Now using (B.1.22)-(B.1.23), we immediately find that there is a non-trivial longitudinal piece ${}^\perp(dx^a) = n^a\,d\sigma$, whereas the transversal piece reads $\underline{dx}^a = dx^a - n^a\,d\sigma$. Obviously $d\sigma$ is purely longitudinal. Hence it is convenient to choose, at an arbitrary point of spacetime, a basis of the cotangent space

$$(d\sigma,\, \underline{dx}^a)\,, \tag{B.1.31}$$

cofol

which is compatible with the foliation given. The coframe (B.1.31) manifestly spans the longitudinal and transversal subspaces of the cotangent space. The corresponding basis of the tangent space reads

$$(n,\, \partial_a)\,. \tag{B.1.32}$$

frfol

This coframe and this frame are anholonomic, in general.

The general decomposition scheme can be applied to the inhomogeneous Maxwell equation (B.1.20). First, we decompose its left hand side. Then the current reads

$$J = {}^\perp J + \underline{J} = -j \wedge d\sigma + \rho\,, \tag{B.1.33}$$

decomin

with

$$j := -J_\perp \qquad \text{and} \qquad \rho := \underline{J}\,. \tag{B.1.34}$$

decomj

The minus sign is chosen in conformity with (B.1.14).

In 3 dimensions, we recover the twisted charge current 2-form j, see (B.1.9), and the twisted charge density 3-form ρ, see (B.1.3). Now charge conservation (B.1.19) can easily be decomposed, too. We substitute (B.1.34) into (B.1.26) for $\Psi = J$.

Then, with the abbreviation (B.1.30), we recover (B.1.12):

$$\dot{\rho} + \underline{d}\, j = 0\,. \qquad \text{(B.1.35)}$$ conti

This, at the same time, gives an exact meaning to the time derivative which we treated somewhat sloppily in Sec.B.1.1. Note that $\underline{d}\,\rho = 0$, as a 4-form in 3 dimensions, represents an identity with no additional information.

Before we turn to the right hand side of (B.1.20), we decompose the excitations according to[4]

$$H = {}^{\perp}H + \underline{H} = d\sigma \wedge \mathcal{H} + \mathcal{D} = -\mathcal{H} \wedge d\sigma + \mathcal{D}\,, \qquad \text{(B.1.36)}$$ decomexi

with the twisted magnetic excitation 1-form $\mathcal{H}$ and the twisted electric excitation 2-form $\mathcal{D}$:

$$\mathcal{H} := H_{\perp} \qquad \text{and} \qquad \mathcal{D} := \underline{H}\,. \qquad \text{(B.1.37)}$$ exi

Everything is now prepared: The longitudinal part of (B.1.20) reads

$${}^{\perp}J = d\sigma \wedge J_{\perp} = {}^{\perp}(dH) = d\sigma \wedge \left(\mathcal{L}_n \underline{H} - \underline{d} H_{\perp}\right) \qquad \text{(B.1.38)}$$ longcur

and the transverse part

$$\underline{J} = \underline{dH} = \underline{d}\; \underline{H}\,. \qquad \text{(B.1.39)}$$ transcur

By means of (B.1.33), (B.1.34), and (B.1.36), the last two equations can be rewritten as

$$\underline{d}\,\mathcal{H} - \dot{\mathcal{D}} = j \qquad \text{(B.1.40)}$$ dH

and

$$\underline{d}\,\mathcal{D} = \rho\,, \qquad \text{(B.1.41)}$$ dD

respectively. We find it remarkable that all we need for recovering the inhomogeneous set of the Maxwell equations (B.1.40),

[4] The historical name of $\mathcal{H}$ is 'magnetic field' and that of $\mathcal{D}$ 'dielectric displacement'.

(B.1.41), see also their vector versions in Eq.(6) of the Introduction, is electric charge conservation in the form (B.1.17) for arbitrary periods – and nothing more. Incidentally, the boundary conditions for $\mathcal{H}$ and $\mathcal{D}$, if required, can also be derived from (B.1.17). Because of the dot, formula (B.1.40) represents an evolution equation, whereas (B.1.41) constitutes a constraint on the initial distributions of $\mathcal{D}$ and ρ.

B.2

Lorentz force density

B.2.1 Electromagnetic field strength

By now we have exhausted the information contained in the axiom of charge conservation. We have to introduce new concepts in order to complete the fundamental structure of electrodynamics. Whereas the excitation $H = (\mathcal{H}, \mathcal{D})$ is linked to the charge current $J = (j, \rho)$, the electric and magnetic field strengths are usually introduced as forces acting on unit charges q at rest or in motion, respectively.

Let us start with a heuristic discussion. We turn first to classical mechanics. Therein the force $\mathcal{F}$ has to be a covector since this is the way it is defined in Lagrangian mechanics, e.g.: $\mathcal{F}_a \sim \partial L/\partial x^a$. In the purely electric case, the force $\mathcal{F}$ acting on a test charge q at rest, in spatial components, reads:

$$\mathcal{F}_a \sim q\, E_a\,. \qquad \text{(B.2.1)}$$ Coul

Thereby we can *define* the electric field strength E in 3-dimensional space. The electric field strength E has 3 independent components exactly as the electric excitation $\mathcal{D}$.

In order to link up mechanics with the rudimentary electrodynamics of the first axiom, we consider a delta-function-like test charge current $J = (j,\, \rho)$ centered around some point with coordinates x^i. Generalizing (B.2.1), the simplest 4-dimensional ansatz for defining the electromagnetic field strength reads:

$$\text{force} \quad \sim \quad \text{field strength} \times \text{charge current}\,. \tag{B.2.2}$$ fieldansatz1

Also in 4 dimensions, the *force* $\sim \partial L/\partial x^i$, with dimension $h\,l^{-1}$, is represented by a *covector*. Here h is an abbreviation for the *dimension of an action*. Accordingly, the ansatz (B.2.2) can be made more precise:

$$f_\alpha = F_\alpha \wedge J\,. \tag{B.2.3}$$ fieldansatz2

The force f_α is a twisted form, since it changes sign under spatial reflections. The dimension of F_α is $[F_\alpha] = [f_\alpha/J] = h\,q^{-1}\,l^{-1}$.

A force cannot possess more than 4 independent components. Since the charge current in (B.2.3) is an twisted 3-form, the only possibility seems to be that f_α is a covector-valued 4-form. Then f_α has 4 components, indeed, and assigns to any 4-volume element the components of a covector:

$$f_\alpha = \frac{1}{4!}\, f_{ijkl\,\alpha}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l\,, \quad f_{ijkl\,\alpha} = f_{[ijkl]\,\alpha}\,. \tag{B.2.4}$$ forcecomp

Since $[\vartheta^{\hat{0}}] = t$, the anholonomic components of f_α, namely $f_{\mu\nu\rho\sigma\,\alpha}$, carry the physical dimension of a force density:

$$[f_{\mu\nu\rho\sigma\,\alpha}] = h\,l^{-1}\,t^{-1}\,l^{-3} = h\,t^{-1}\,l^{-4} \overset{SI}{=} J\,m^{-4} = N\,m^{-3}\,. \tag{B.2.5}$$ dimen

Thus we have identified in (B.2.3) the force density f_α. As a consequence, the field F_α turns out to be an untwisted covector-valued 1-form $F_\alpha = F_{\alpha\beta}\,\vartheta^\beta$, with the coframe ϑ^β. As such, it would have 16 independent components in general. However, we know already that the electric field strength E has 3 independent components. If we expect the analogous to be true for the magnetic field strength B, then $F_{\alpha\beta}$ should carry 6 independent components at most. In other words, it must be antisymmetric: $F_{\alpha\beta} = -F_{\beta\alpha}$. Accordingly, the electromagnetic field strength turns out to be a 2-form $F = \frac{1}{2}\, F_{\alpha\beta}\, \vartheta^\alpha \wedge \vartheta^\beta$ and $F_\alpha = e_\alpha \rfloor F = F_{\alpha\beta}\, \vartheta^\beta$, with the frame e_α.

With this assumption, the force equation (B.2.3) for a test charge current reads[1] $f_\alpha = (e_\alpha \rfloor F) \wedge J$. Accordingly, we defined the field strength F as incorporation of possible forces acting in an electromagnetic field on test charges thereby relating the mechanical notion of a force to the electromagnetic state around electric charges and currents. In the first axiom, we consider the active role of charge that creates the excitation field; here we study its passive role, namely which forces act on it in an electromagnetic environment.

B.2.2 Second axiom relating mechanics and electrodynamics

We have then

$$f_\alpha = (e_\alpha \rfloor F) \wedge J \qquad \text{(second axiom)}. \tag{B.2.6}$$ axiom2

The *untwisted electromagnetic field strength 2-form* F carries the dimension $[F] = h\, q^{-1}$. This equation for the Lorentz force density yields an *operational* definition of the electromagnetic field strength F and represents our second axiom of electrodynamics. Observe that (B.2.6), like (B.1.17), is an equation which is free from metric and connection, it is defined on any 4-dimensional differentiable manifold. Since $F \wedge J$ is a 5-form, (B.2.6) can alternatively be written as $f_\alpha = -F \wedge (e_\alpha \rfloor J)$.

A decomposition of the 2-form F into 'time' and 'space' pieces, according to

$$F = {}^\perp F + \underline{F} = -d\sigma \wedge E + B = E \wedge d\sigma + B\,, \tag{B.2.7}$$ decompF

yields, in 3 dimensions, the untwisted electric field strength 1-form E and the untwisted magnetic field strength[2] 2-form B:

[1]The Lorentz force acting on a particle with charge e and with velocity v, turns out to be $\mathcal{F} = ev \rfloor F$. This formula can be derived from our second axiom.

[2]The historical names are 'electric field' for E and 'magnetic induction' for B.

$$E := -F_\perp \qquad \text{and} \qquad B := \underline{F}\,, \tag{B.2.8}$$

eb

Clearly then, the electric line tension (electromotive force or voltage) $\int\limits_{\Omega_1} E$ and the magnetic flux $\int\limits_{\Omega_2} B$ must play a decisive role in Maxwell's theory. Hence, as building blocks for laws governing the electric and magnetic field strength, we have only the electric line tension and the magnetic flux at our disposal.

The Lorentz force density f_α, as a 4-form, that is, as a form of maximal rank, is purely longitudinal with respect to the normal vector n. Thus only its longitudinal piece $(f_\alpha)_\perp$, a 3-form, survives. It turns out to be

$$\begin{aligned} k_\alpha := \left(f_\alpha\right)_\perp &= n \rfloor \left[(e_\alpha \rfloor F) \wedge J\right] \\ &= (e_\alpha \rfloor E) \wedge J + (e_\alpha \rfloor F) \wedge j \\ &= \rho \wedge (e_\alpha \rfloor E) + j \wedge (e_\alpha \rfloor B) \\ &\qquad - j \wedge E \wedge (e_\alpha \rfloor d\sigma)\,. \end{aligned} \tag{B.2.9}$$

longLor

If we now display its time and space components, we find

$$k_{\hat{0}} = -j \wedge E\,, \tag{B.2.10}$$

con0

$$k_a = \rho \wedge (e_a \rfloor E) + j \wedge (e_a \rfloor B)\,. \tag{B.2.11}$$

con1

The time component $k_{\hat{0}}$ represents the *electric power density*, the space components k_a the 3-dimensional Lorentz force density. Note that $E_a = e_a \rfloor E$ are the components of the ordinary 3-covector of the electric field strength. However, for the magnetic field strength we have $e_a \rfloor B = e_a \rfloor \left(\mathcal{B}^b\, \hat{\underline{\epsilon}}_b\right) = \mathcal{B}^b\, \hat{\underline{\epsilon}}_{ba}$, where $\mathcal{B}^c = \frac{1}{2}\epsilon^{abc} B_{ab}$ is equivalent to the components of the 3-vector density of the conventional magnetic field strength.

If there is an electromagnetic field configuration such that the Lorentz force density vanishes, $f_\alpha = 0$, we call it a *force-free electromagnetic* field:

$$(e_\alpha \rfloor F) \wedge dH = 0\,. \tag{B.2.12}$$

ffree1

Here we substituted already the inhomogeneous Maxwell equation.

In plasma physics such configurations play a decisive role if restricted purely to the magnetic field. We 1+3 decompose (B.2.12). A look at (B.2.10) and (B.2.11) shows that for the magnetic field only the space components (B.2.11) of the Lorentz force density matter:

$$k_a = \underline{d}\,\mathcal{D} \wedge (e_a \rfloor E) - \dot{\mathcal{D}} \wedge (e_a \rfloor B) + \underline{d}\,\mathcal{H} \wedge (e_a \rfloor B)\,. \tag{B.2.13}$$

If the electric excitation and its time derivative vanish, $\mathcal{D} = 0$, $\dot{\mathcal{D}} = 0$ — clearly a frame dependent, i.e., non-covariant statement — then we find for the *force-free magnetic* field[3]

$$(e_a \rfloor B) \wedge \underline{d}\,\mathcal{H} = 0 \tag{B.2.14}$$

ffree2

or, in components,

$$\mathcal{B}^b\, \partial_{[a}\mathcal{H}_{b]} = 0\,. \tag{B.2.15}$$

ffree3

B.2.3 The first three invariants of the electromagnetic field

The first axiom supplied us with the fields J (twisted 3-form) and H (twisted 2-form) and the second axiom, additionally, with F (untwisted 2-form). Algebraically we can construct therefrom the so-called first invariant of the electromagnetic field,

$$I_1 := F \wedge H\,, \qquad [I_1] = h\,. \tag{B.2.16}$$

firstinv

It is a twisted 4-form or, equivalently, a scalar density of weight +1 with one independent component. Clearly, I_1 could qualify as a Lagrange 4-form: It is a twisted form and it has the appropriate dimension.

Furthermore, a second invariant can be assembled,

$$I_2 := F \wedge F\,, \qquad [I_2] = (h/q)^2\,, \tag{B.2.17}$$

secondinv

[3]See Lüst & Schlüter [22]. However, they as well as later authors put $B = \mu_0\, {}^\star\mathcal{H}$ right away and start with $\vec{\mathcal{H}} \times \nabla \times \vec{\mathcal{H}} = 0$.

an untwisted 4-form with a somewhat strange dimension, and a third one,

$$I_3 := H \wedge H\,, \qquad [I_3] = q^2\,, \tag{B.2.18}$$ thirdinv

equally an untwisted form.

In order to get some insight into the meaning of these 4-forms, we substitute the (1+3)-decompositions (B.1.36) and (B.2.7):

$$I_1 = F \wedge H = \quad d\sigma \wedge (B \wedge \mathcal{H} - E \wedge \mathcal{D})\,, \tag{B.2.19}$$ FH

$$I_2 = F \wedge F = -2\, d\sigma \wedge\ B \wedge E, \tag{B.2.20}$$ secondinv1

$$I_3 = H \wedge H = \quad 2\, d\sigma \wedge \mathcal{H} \wedge \mathcal{D}\,. \tag{B.2.21}$$ HH

If for an electromagnetic field configuration the first invariant vanishes, then we find,

$$I_1 = 0 \qquad \text{or} \qquad \frac{1}{2} B \wedge \mathcal{H} = \frac{1}{2} E \wedge \mathcal{D}\,. \tag{B.2.22}$$ dK=0

As we will see in (B.5.52), this means that the magnetic energy density equals the electric energy density. Similarly, for the second invariant, we have

$$I_2 = 0 \qquad \text{or} \qquad B \wedge E = 0\,. \tag{B.2.23}$$ dC=0

Then the electric field strength 1-form can be called "parallel" to the magnetic strength field 2-form. Analogously, for the vanishing third invariant,

$$I_3 = 0 \qquad \text{or} \qquad \mathcal{H} \wedge \mathcal{D} = 0\,, \tag{B.2.24}$$ thirdi

the excitations are "parallel" to each other. It should be understood that the characterizations $I_1 = 0$ etc. are (diffeomorphism and frame) invariant statements about electromagnetic field configurations.

This is as far as we can go by algebraic manipulations. If we differentiate, we can construct dJ, dH, dF. The first two expressions are known, $dJ = 0$ and $dH = J$, whereas the last one, dF, is left open so far. We will turn to it in the next chapter.

Table B.2.1: Invariants of the electromagnetic field in 4D.

Invariant	**dimension**	**un-/twisted**	**name**
$I_1 = F \wedge H$	h	twisted	$\sim$ Lagrangian
$I_2 = F \wedge F$	$(h/q)^2$	untwisted	Chern 4-form
$I_3 = H \wedge H$	q^2	untwisted	...
$I_4 = A \wedge J$	h	twisted	coupling term in Lagr.

Let us collect our results in a table:
The fourth invariant I_4 and the names will be explained in Sec. B.3.3. The I_i's are 4-forms, respectively, with 1 independent component each. We call them invariants. With the diamond operator $^\diamond$, the dual with respect to the Levi-Civita epsilon (see (A.1.80) at the end of Sec A.1.9), we can attach to each 4-form I_i a (metric-free) scalar *density* $^\diamond I_i$.

B.3

Magnetic flux conservation

B.3.1 Third axiom

The spacetime manifold, which underlies our consideration and which has been defined at the beginning of Sec. B.1.2, is equipped with the property of an orientation. Then we can integrate the untwisted 2-form F in 4 dimensions over a 2-dimensional surface. Since F is a 2-form, the simplest invariant statement, which comes to mind, would read

$$\oint_{C_2} F = 0\,, \qquad \partial C_2 = 0 \qquad \text{(third axiom)}\,, \qquad \text{(B.3.1)}$$ axiom3

for any closed 2-dimensional submanifold $C_2 \subset X_4$. Indeed, this is the axiom we are looking for.

It is straightforward to find supporting evidence for (B.3.1) by using the decomposition (B.2.7). Faraday's induction law results from (B.3.1) if one chooses the 2-dimensional surface as $C_2' = \partial\Omega'_3$, with $\Omega'_3 = [\sigma_0, \sigma] \times \Omega'_2$, where Ω'_2 is represented, in Fig.B.3.1, by a line in h_{σ_0}, i.e., it is transversal to the vector

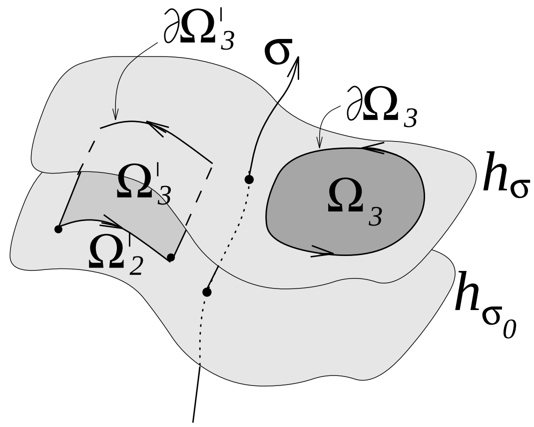

Figure B.3.1: Different 2-dimensional periods of the flux integral.

field n:

$$\oint_{\partial\Omega_2} E + \frac{d}{d\sigma}\int_{\Omega_2} B = 0\,. \qquad \text{(B.3.2)}$$ induction

What is usually called the law of the absence of magnetic charge, we find by again choosing the 2-dimensional surface C_2 in (B.3.1) as a boundary of a 3-dimensional submanifold Ω_3 which is lying in one of the folia h_σ (see Fig.B.3.1):

$$\oint_{\partial\Omega_3\subset h_\sigma} B = 0\,. \qquad \text{(B.3.3)}$$ nomono

The proofs are analogous to the one given in (B.1.13). They will be given below on a differential level.

Magnetic flux $\int_{\Omega_2} B$ and its conservation is of central importance to electrodynamics. At low temperatures, certain materials can become superconducting, i.e., they lose their electrical resistance. At the same time, if they are exposed to an external (sufficiently weak) magnetic field, the magnetic field is expelled from their interior except for a thin layer at their surface (Meissner-Ochsenfeld effect). In the case of a superconductor of type II, Niobium, for example, provided the external field is higher than a certain critical value, *quantized* mag-

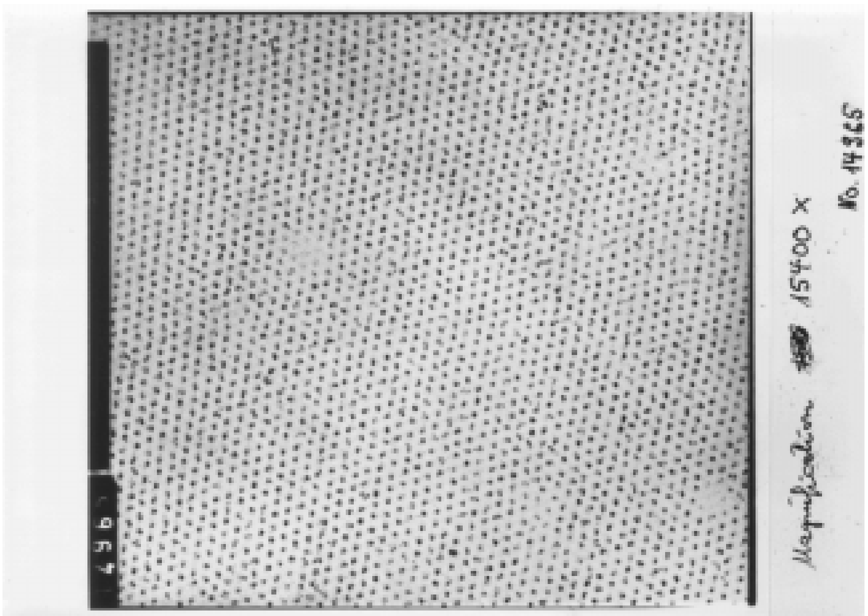

Figure B.3.2: *Direct observation of individual flux lines in type II superconductors according to Essmann & Träuble [7, 8].* The image shown here belongs to a small superconducting Niobium disc (diameter 4 mm, thickness 1 mm) which, at a temperature of 1.2 K, was exposed to an external magnetic excitation of $\mathcal{H} = 78\ kA/m$. At the surface of the disc the flux lines were decorated by small ferromagnetic particles which were fixed by a replica technique. Eventually the replica was observed by means of an electron microscope. The parameter of the flux-line lattice was 170 nm (courtesy of U. Essmann).

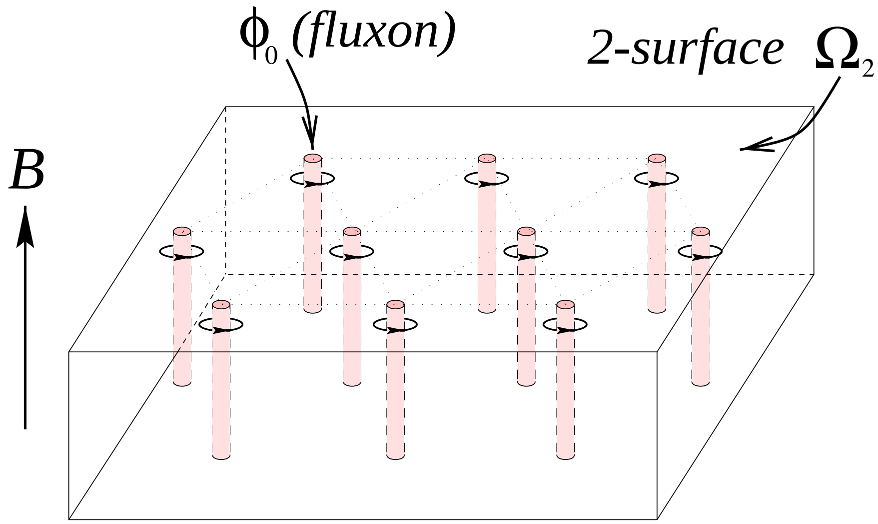

Figure B.3.3: Sketch of an Abrikosov lattice in a type II superconductor in 3-dimensional space.

netic flux lines carrying a flux quantum of[1] $\Phi_0 := \mathrm{h}/(2e) \overset{SI}{\approx} 2.068 \times 10^{-15}$ *Weber* can penetrate from the surface of the superconductor and can build up a triangular lattice, an Abrikosov lattice. A cross section of such a flux line lattice is depicted in Fig. B.3.2, a schematic view provided in Fig. B.3.3. What is important for us is that, at least under certain circumstances, single quantized flux lines can be *counted* and that they behave like a *conserved* quantity, i.e., they migrate but are not spontaneously created nor destroyed. The counting argument supports the view that magnetic flux is determined in a metric-free way, the migration argument, even if this is somewhat indirect, that the flux is conserved. The computer simulation of the magnetic field of the Earth in Fig. B.3.4 may give an intuitive feeling that the magnetic lines of force, which can be understood as unquantized magnetic flux lines, are close to the induction law and, at the same time, close to our visual perception of magnetic field configurations.

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[1] Here h is Planck's constant and e the elementary charge.

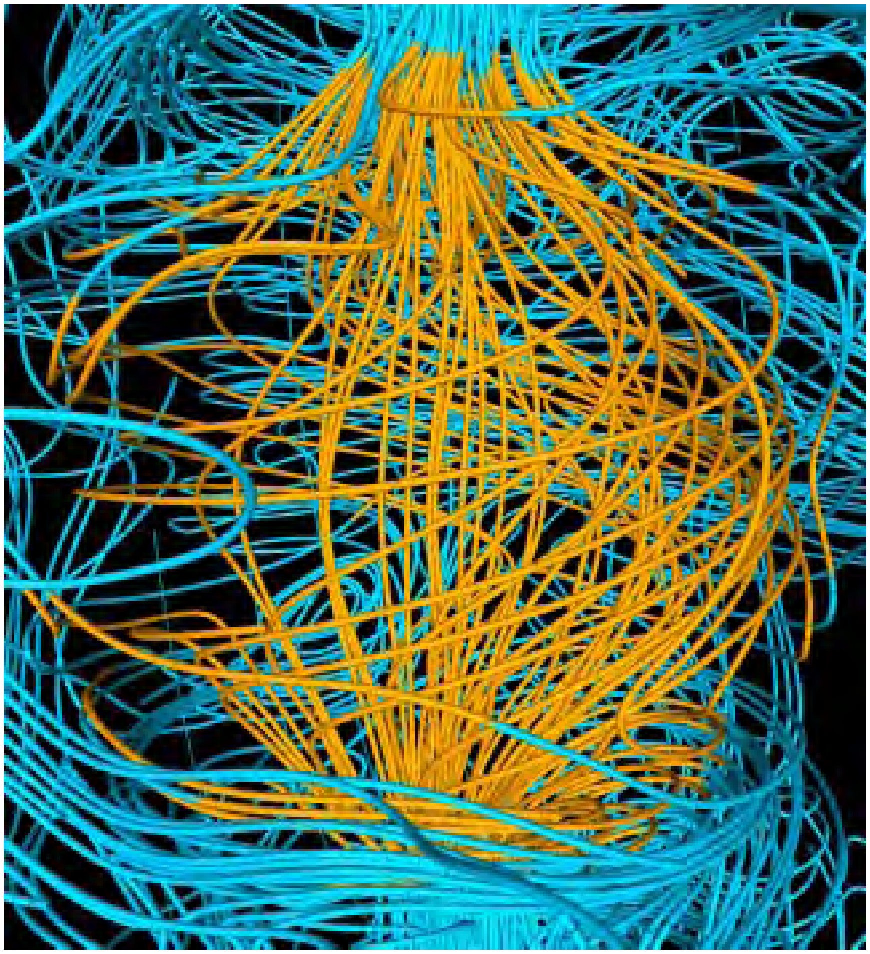

Figure B.3.4: Figure of G.A. Glatzmaier: Snapshot of magnetic lines of force in the core of our computer simulated Earth. Lines in gold (blue) where they are inside (outside) of the inner core. The axis of rotation is vertical in this image. The field is directed inward at the inner core north pole (top) and outward at the south pole (bottom); the maximum magnetic intensity is about 30 mT; see [13] and [34].

B.3.2 Electromagnetic potential

In (B.3.1) we specialize to the case when C_2 is a 2-boundary $C_2 = \partial\Omega_3$ of an arbitrary 3-dimensional domain Ω_3. Then, by Stokes' theorem, we find

$$\int\limits_{\Omega_3} d\,F = 0\,. \tag{B.3.4}$$

intdF

Since Ω_3 can be chosen arbitrarily, the electromagnetic field strength turns out to be a closed form:

$$d\,F = 0\,. \tag{B.3.5}$$

dF0

This is the 4-dimensional version of the set of the homogeneous Maxwell equations.

The axiom (B.3.1) now tells that all periods of F are zero. Consequently the field strength is an exact form

$$F = d\,A\,. \tag{B.3.6}$$

Fclosed

Eq.(B.3.5) is implied by (B.3.6) because of $dd = 0$. However, the inverse statement that (B.3.5) implies (B.3.6) does not hold in a global manner unless the conditions for the first de Rham theorem are met.

The untwisted electromagnetic potential 1-form A has the dimension of $h\,q^{-1}$. Decomposed in 'time' and 'space' pieces, it reads

$$A = \varphi\,d\sigma + \mathcal{A}\,, \tag{B.3.7}$$

decomA1

with

$$\varphi := A_\perp \qquad \text{and} \qquad \mathcal{A} := \underline{A}\,. \tag{B.3.8}$$

decomA2

Here we recover the familiar 3-dimensional scalar and covector potentials φ and $\mathcal{A}$, respectively. The potential is only determined up to a closed 1-form

$$A \longrightarrow A + \chi\,, \qquad d\,\chi = 0\,, \tag{B.3.9}$$

gaugeA

a fact which has far-reaching consequences for the quantization of the electromagnetic field. We decompose (B.3.6) and find straightforwardly

$$
{}^{\perp}F = {}^{\perp}(dA) \qquad \text{or} \qquad E = \underline{d}\varphi - \dot{\mathcal{A}} \tag{B.3.10}
$$

decomA3

and

$$
\underline{F} = \underline{dA} \qquad \text{or} \qquad B = \underline{d}\mathcal{A}\,. \tag{B.3.11}
$$

decomA4

A decomposition of (B.3.5), by means of (B.1.26) and (B.2.8), yields the homogeneous set of Maxwell's equations,

$$
{}^{\perp}(dF) = d\sigma \wedge \left(\mathcal{L}_n \underline{F} - \underline{d}F_{\perp}\right) = 0\,, \qquad \underline{dF} = 0\,, \tag{B.3.12}
$$

maxinhom

or, in the conventional 3-dimensional notation,

$$
\underline{d}\,E + \dot{B} = 0 \tag{B.3.13}
$$

maxinhom3d1

and

$$
\underline{d}\,B = 0\,, \tag{B.3.14}
$$

maxinhom3d2

respectively. If we integrate these equation over a 2- or a 3-dimensional volume and apply the Stokes theorem, we find (B.3.2) and (B.3.3), q.e.d.. Again, in analogy to the inhomogeneous equations, (B.3.13) and (B.3.14) represent an equation of motion and a constraint respectively.

B.3.3 Abelian Chern-Simons and Kiehn 3-forms

For the electromagnetic theory, we got now the 1-form A as a new building block. This has an immediate consequence for the second invariant, namely

$$
I_2 = (dA) \wedge F = d(A \wedge F) = d(A \wedge dA)\,, \tag{B.3.15}
$$

chern1

since $dF = 0$. In other words, I_2, the so-called Abelian *Chern* 4-form, is an *exact* form and, accordingly, cannot be used as a non-trivial Lagrangian. We read off from (B.3.15) the untwisted Abelian *Chern-Simons* 3-form

$$C_A := A \wedge F\,, \qquad [C_A] = (h/q)^2\,, \tag{B.3.16}$$ chern

which has a certain topological meaning. Also in all other dimensions, with $n \geq 3$, the Abelian Chern-Simons form is represented by a 3-form. If we $(1+3)$-decompose C_A, we find

$$C_A = \mathcal{A} \wedge B + d\sigma \wedge (\varphi\, B + \mathcal{A} \wedge E)\,, \tag{B.3.17}$$ chern13

which includes the so-called magnetic helicity[2] $\mathcal{A} \wedge B$, likewise a 3-form (however in 3D) with one independent component. Obviously,

$$dC_A = F \wedge F\,. \tag{B.3.18}$$ dc

Consequently, even though C_A is not gauge invariant, its differential dC_A is gauge invariant.

What we just did to the second invariant, we can now implement, in an analogous way, for the first invariant:

$$I_1 = (dA) \wedge H = d(A \wedge H) + A \wedge J\,. \tag{B.3.19}$$ kiehn1

We define the *twisted Kiehn* 3-form[3]

$$K := A \wedge H\,, \qquad [K] = h\,. \tag{B.3.20}$$ kiehn

It carries the dimension of an action. In n dimensions, it would be an $(n-1)$-form — in sharp contrast to the Abelian Chern-Simons 3-form. If we decompose K into 1+3, we find

$$K = \mathcal{A} \wedge \mathcal{D} + d\sigma \wedge (-\varphi\, \mathcal{D} + \mathcal{A} \wedge \mathcal{H})\,. \tag{B.3.21}$$ kiehn13

The 2-form $\mathcal{A} \wedge \mathcal{H}$ is the purely magnetic piece of K in 3D. By differentiating K, or directly from (B.3.19), we find

$$dK = F \wedge H - A \wedge J\,. \tag{B.3.22}$$ dk

---

[2] For *magnetic helicity*, see Moffatt [27], Marsh [23, 24], and Rañada [30, 31, 32].

[3] See Kiehn and Pierce [20, 18, 19].

Even though the Kiehn 3-form changes under a gauge transformation $A \to A + d\psi$, its exterior derivative dK, provided we are in the free-field region with $J = 0$, is gauge-*in*variant, as can be seen in (B.3.22).

¿From (B.3.22), we can read off the new twisted electromagnetic interaction 4-form

$$I_4 := A \wedge J\,, \qquad [I_4] = h\,. \tag{B.3.23}$$

4inv

In decomposes according to

$$A \wedge J = d\sigma \wedge (\varphi\,\rho + \mathcal{A} \wedge j)\ . \tag{B.3.24}$$

AJ

Because of $dJ = 0$, it is gauge invariant up to an exact form:

$$I_4 \quad \longrightarrow \quad I_4 + (d\psi) \wedge J = I_4 + d(\psi \wedge J)\,. \tag{B.3.25}$$

AJgauge

Accordingly, I_4, besides I_1, also qualifies as a piece of an electrodynamic Lagrangian 4-form, since it is twisted, of dimension h, and gauge invariant (up to an irrelevant exact form). Note, however, that dK in (B.3.22), as an exact form, cannot feature as a Lagrangian in 4D even though both pieces on the right hand side of (B.3.22) for themselves have the correct behavior. In other word, the relative factor beween I_1 and I_4 is inappropriate for a Lagrange 4-form.

Let us put our results together. If we start with the 2-form H and differentiate and multiply, then we find the sequence $(H, dH, H \wedge H)$. We cannot go any further since forms with a rank $p > 4$ will vanish identically. Similarly, for the 1-form A, we have $(A, dA, A \wedge dA, dA \wedge dA)$, and, if we mix both, then the sequence $(A \wedge H, dA \wedge H, A \wedge dH)$ emerges. Note that $A \wedge A \equiv 0$ since A is a 1-form. In this way, we create the new 3-forms K, see (B.3.20), and C_A, see (B.3.16):
By the same token, the four different invariants I_i of the electromagnetic field arise as 4-forms, see Table B.2.1 on page @@@.

B.3.4 Measuring the excitation

The electric excitation $\mathcal{D}$ can be operationally defined by using the Gauss law (B.1.41). Put at some point P in free space

Table B.3.1: Three-forms of the electromagnetic field in in 4D.

3-form	**dimension**	**un-/twisted**	**name**
$K = A \wedge H$	h	twisted	Kiehn
$C_A = A \wedge F$	$(h/q)^2$	untwisted	Chern-Simons
J	q	untwisted	electric current

two small, thin, and electrically conducting (metal) plates of arbitrary shape with insulating handles ("Maxwellian double plates"), see Fig.B.3.5. Suppose we choose local coordinates (x^1, x^2, x^3) in the neighborhood of P such that $P = (0, 0, 0)$. The vectors ∂_a, with $a = 1, 2, 3$, span the tangent space at P. Press the plates together at P, and orient them in such a way that their common boundary is given by the equation $x^3 = 0$ (and thus (x^1, x^2) are the local coordinates on each plate's surface). Separate the plates and measure the net charge $\mathcal{Q}$ and the area S of that plate for which the vector ∂_3 points outwards from its boundary surface. Determine experimentally $\lim_{S\to 0} \mathcal{Q}/S$ as well as possible. Then

$$\mathcal{D}_{12} = \lim_{S\to 0} \frac{\mathcal{Q}}{S}\,. \qquad \text{(B.3.26)}$$

dsurf1

Technically, one can realize the limiting process by constructing the plates in the form of parallelograms with the sides $a\,\partial_1$ and $a\,\partial_2$. Then (B.3.26) is replaced by

$$\mathcal{D}_{12} = \lim_{a\to 0} \frac{\mathcal{Q}}{a^2}\,, \qquad \text{(B.3.27)}$$

dsurf1h

Repeat this measurement with the two other possible orientations of the plates (be careful with choosing one of the two plates in accordance with the orientation prescription of above). Then, similarly, one finds $\mathcal{D}_{23}$ and $\mathcal{D}_{31}$. Thus finally the electric excitation is measured:

$$D = \frac{1}{2}\mathcal{D}_{ab} dx^a \wedge dx^b = \mathcal{D}_{12} dx^1 \wedge dx^2 + \mathcal{D}_{23} dx^2 \wedge dx^3 + \mathcal{D}_{31} dx^3 \wedge dx^1\,. \qquad \text{(B.3.28)}$$

dsurf2

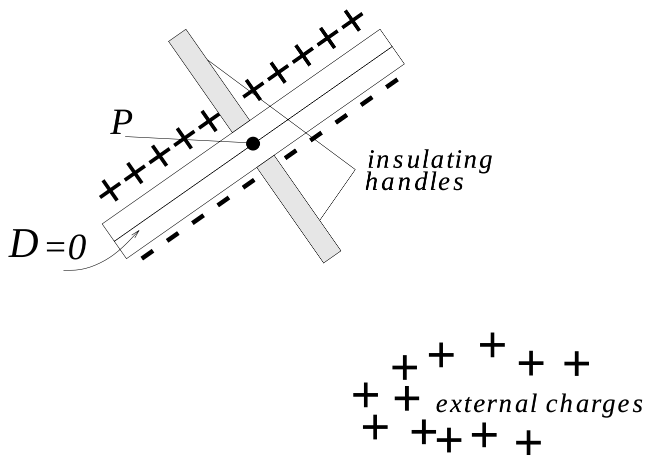

Figure B.3.5: Measurement of $\mathcal{D}$ at a point P.

The dimension of $\mathcal{D}$ is that of a *charge*, i.e. $[\mathcal{D}] = q$, for its components we have $[\mathcal{D}_{ab}] = q\, l^{-2}$.

Let us prove the correctness of this prescription. The double plates are assumed to be ideal *conductors*. Therefore the electric field E inside the conductor has to vanish:

$$E\Big|_{\text{inside}} = 0. \qquad \text{(B.3.29)}$$ Ein

According to (B.2.6), the electric field E is uniquely defined. Thus the vanishing of E defines a unique electrodynamical state in the conductor. And for the electric excitation $\mathcal{D}$, guided by experience,[4] we assume that, in turn, it vanishes, too:

$$\mathcal{D}\Big|_{\text{inside}} = 0. \qquad \text{(B.3.30)}$$ Din

This is a rudiment of the spacetime relation that links F and H and which makes the excitation field $\mathcal{D}$ unique. For that reason we had postponed this discussion until now, since the knowledge of the notion of the field strength E is necessary for it.

[4]The only really appropriate discussion on the definitions of $\mathcal{D}$ and E, which we know of, was given by Pohl [28].

By (B.3.30), we *selected* from the allowed class (B.1.21) of the excitations that $\mathcal{D}$ which will be measured by the double plates.

In a more general setting, let us consider the electromagnetic excitation $\mathcal{D}$ near a 2-dimensional boundary surface $S \subset h_\sigma$ which separates two parts of space filled with two different types of matter. Choose a point $P \in S$. Introduce local coordinates (x^1, x^2, x^3) in the neighborhood of P in such a way that $P = (0,0,0)$, while (x^1, x^2) are the coordinates on S, see Fig.B.3.6. Let V be a 3-dimensional domain which is half (denoted V_1) in one medium and half (V_2) in another one, with the two halves $V_{1,2}$ spanned by the triples of vectors $(\partial_1, \partial_2, a\partial_3)$ and $(\partial_1, \partial_2, -a\partial_3)$, respectively. For definiteness, we assume that ∂_3 points from medium 1 to medium 2. Now we consider the Gauss law (B.1.41) in this domain. Integrate (B.1.41) over V, and take the limit $a \to 0$. The result is

$$\int\limits_{\Delta S_{12}} \mathcal{D}_{(2)} - \int\limits_{\Delta S_{12}} \mathcal{D}_{(1)} = \lim_{a\to 0} \int\limits_{V} \rho\,, \qquad \text{(B.3.31)}$$

difD1

where $\mathcal{D}_{(1)}$ and $\mathcal{D}_{(2)}$ denote the values of the electric excitations in the medium 1 and 2, respectively, while ΔS_{12} is a piece of S spanned by (∂_1, ∂_2).

Despite the fact that the volume V clearly goes to zero, the right-hand side of (B.3.31) is nontrivial when there is a *surface charge* exactly on the boundary S. Mathematically, in local coordinates (x^1, x^2, x^3), this can be described by the δ-function structure of the charge density:

$$\rho = \rho_{123}\, dx^1 \wedge dx^2 \wedge dx^3 = \delta(x^3)\, \widetilde{\rho}_{12}\, dx^1 \wedge dx^2 \wedge dx^3\,. \qquad \text{(B.3.32)}$$

delrho

Substituting (B.3.32) into (B.3.31), we then find

$$\int\limits_{\Delta S_{12}} \mathcal{D}_{(2)} - \int\limits_{\Delta S_{12}} \mathcal{D}_{(1)} = \int\limits_{\Delta S_{12}} \widetilde{\rho}\,, \qquad \text{(B.3.33)}$$

difD2

where $\widetilde{\rho} = \widetilde{\rho}_{12}\, dx^1 \wedge dx^2$ is the *2-form of the electric surface charge density.* Since P and ΔS_{12} are arbitrary, we conclude

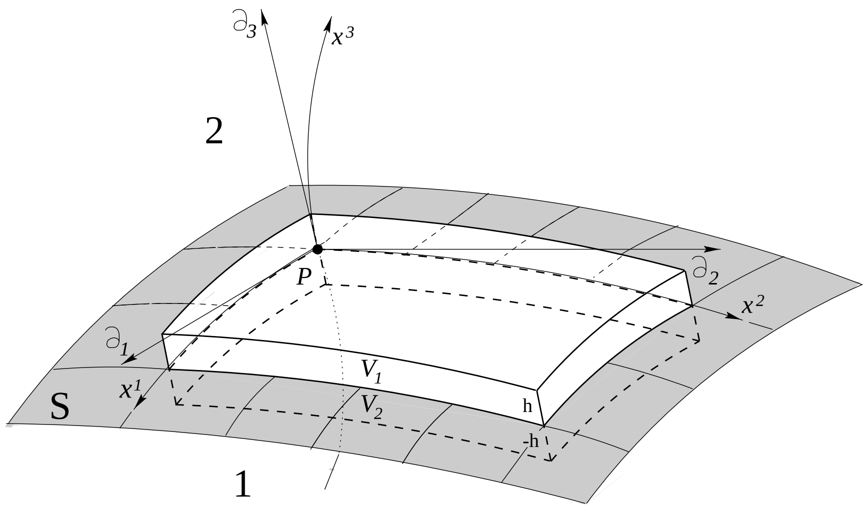

Figure B.3.6: Electric excitation on the boundary between two media.

that on the separating surface S between two media, the electric excitation satisfies

$$\mathcal{D}_{(2)}\Big|_{\rm S} - \mathcal{D}_{(1)}\Big|_{\rm S} = \widetilde{\rho}. \qquad \text{(B.3.34)}$$ DonS

Returning to the measurement process with plates, we have $\mathcal{D}_{(1)} = 0$ inside an ideal conductor, and (B.3.34) justifies the definition of the electric excitation as the charge density on the surface of the plate (B.3.26). Thereby we recognize that the electric excitation $\mathcal{D}$ *is*, by its very definition, the ability to separate charges on (ideally conducting) double plates.

A similar result holds for the magnetic excitation $\mathcal{H}$. Analogously to the small 3-dimensional domain V, let us consider a 2-dimensional domain Σ which is half (Σ_1) in one medium and half (Σ_2) in another one, with the two halves $\Sigma_{1,2}$ spanned by the vectors $(l, a\partial_3)$ and $(l, -a\partial_3)$, respectively. Here $l = l^1\partial_1 + l^2\partial_2$ is an arbitrary vector tangent to S at P. Integrating the Maxwell equation (B.1.40) over Σ, and taking the limit $a \to 0$, we find

$$\int_l \mathcal{H}_{(2)} - \int_l \mathcal{H}_{(1)} = \lim_{a\to 0} \int_\Sigma j. \tag{B.3.35}$$

difH1

The second term on left-hand side of (B.1.40) has a zero limit, because $\dot{\mathcal{D}}$ is continuous and finite in the domain of a loop which contracts to zero. However, the right-hand side of (B.3.35) produces a nontrivial result when there are surface currents flowing exactly on the boundary surface S. Analogously to (B.3.32), this is described by

$$j = j_{13} dx^1 \wedge dx^3 + j_{23} dx^2 \wedge dx^3 = \delta(x^3)\, \widetilde{j}_a \, dx^a \wedge dx^3. \tag{B.3.36}$$

delj

Substituting (B.3.36) into (B.3.35), we find

$$\int_l \mathcal{H}_{(2)} - \int_l \mathcal{H}_{(1)} = \int_l \widetilde{j}, \tag{B.3.37}$$

and, since l is arbitrary, eventually:

$$\mathcal{H}_{(2)}\Big|_{\mathrm{S}} - \mathcal{H}_{(1)}\Big|_{\mathrm{S}} = \widetilde{j}. \tag{B.3.38}$$

HonS

Thus we see that on the boundary surface between the two media the magnetic excitation is directly related to the *1-form of the electric surface current density* $\widetilde{j}$.

To measure $\mathcal{H}$, following a suggestion of M. Zirnbauer,[5] we can use the Meissner effect. Because of this effect, the magnetic field B is driven out from the superconducting state. Take a thin superconducting wire (since the wire is pretty cold, perhaps around 10 K, taking is not to be understood too literally), and put it at the point P where you want to measure $\mathcal{H}$ (see Fig.B.3.7(a)). Because of the Meissner effect, the magnetic field B and, if we assume again that $B = 0$ implies $\mathcal{H} = 0$, the excitation $\mathcal{H}$ is expelled from the superconducting region (apart from

[5] See his lecture notes [40], compare also Ingarden and Jamiołkowski [16].

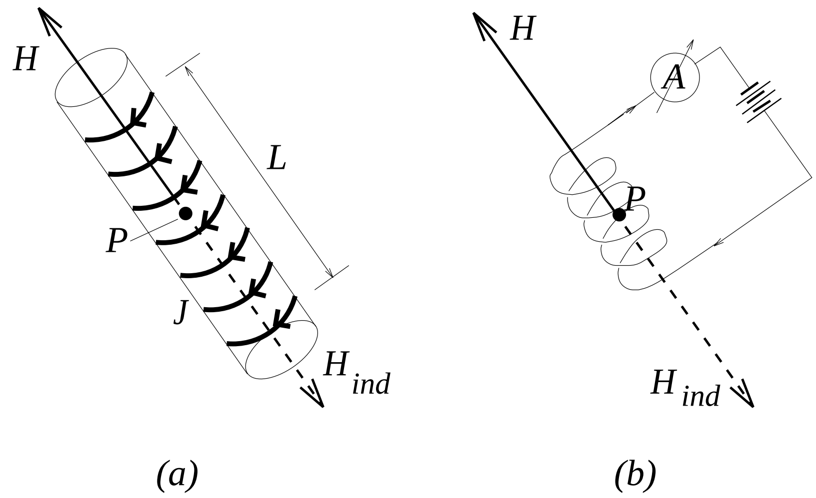

Figure B.3.7: Measurement of $\mathcal{H}$ at a point P.

a thin surface layer of some 10 nm, where $\mathcal{H}$ can penetrate). According to the Oersted-Ampère law (B.1.40) $\underline{d}\mathcal{H} = j$ – we assume quasi-stationarity in order to be permitted to forget about $\dot{\mathcal{D}}$ – the compensation of $\mathcal{H}$ at P can be achieved by surface currents $\mathcal{J}$ flowing around the superconducting wire. These induced surface currents $\mathcal{J}$ are of such a type that they generate an $\mathcal{H}_{\text{ind}}$ which compensates the $\mathcal{H}$ to be measured: $\mathcal{H}_{\text{ind}} + \mathcal{H} = 0$. We have to change the angular orientation such that we eventually find the maximal current $\mathcal{J}_{\text{max}}$. Then, if the x^3-coordinate line is chosen tangentially to the "maximal orientation", we have

$$\mathcal{H}_3 = \lim_{L \to 0} \frac{\mathcal{J}_{\text{max}}}{L}\,, \qquad \text{(B.3.39)}$$ max

where L is the length parallel to the wire axis transverse to which the current has been measured. Accordingly, we find

$$\mathcal{H} = \mathcal{H}_3\, dx^3\,. \qquad \text{(B.3.40)}$$ H3

Clearly the dimension of $\mathcal{H}$ is that of an *electric current*: $[\mathcal{H}] = q\,t^{-1} \overset{SI}{=} A$ and $[\mathcal{H}_a] = q\,l^{-1}\,t^{-1} \overset{SI}{=} A/m$.

If you dislike this thought experiment, you can take a real small *test coil* at P, orient it suitably, and read off the corresponding maximal current at a galvanometer as soon as the effective $\mathcal{H}$ vanishes (see Fig.B.3.7(b)). Multiply this current with the winding number of the coil and find $\mathcal{J}_{\max}$. Divide by the length of the test coil, and you are back to (B.3.39). Whether $\mathcal{H}$ is really compensated for, you can check with a magnetic needle which, in the field-free region, should be in an indifferent equilibrium state.

In this way, we can build up the excitation $H = \mathcal{D} - \mathcal{H} \wedge d\sigma$ as a measurable electromagnetic quantity in its own right. The excitation H, together with the field strength F, we will call the "electromagnetic field".

B.4

Basic classical electrodynamics summarized, example

B.4.1 Integral version and Maxwell's equations

We are now in a position to summarize the fundamental structure of electrodynamics in a few lines. According to (B.1.17), (B.2.6), and (B.3.1), the three axioms on a connected, Hausdorff, paracompact, and oriented spacetime read, for any C_3 and C_2 with $\partial C_3 = 0$ and $\partial C_2 = 0$:

$$\oint_{C_3} J = 0\,, \qquad f_\alpha = (e_\alpha \rfloor F) \wedge J\,, \qquad \oint_{C_2} F = 0\,. \qquad \text{(B.4.1)}$$ 3axioms

The first axiom reigns matter and its conserved electric charge, the second axiom links the notion of that charge and the concept of a mechanical force to an operational definition of the electromagnetic field strength. The third axiom determines the flux of the field strength as source-free.

In Part C we will learn that a metric of spacetime brings in the temporal and spatial distance concepts and a linear connection the inertial guidance field (and thereby parallel displacement). Since the metric of spacetime represents Einstein's *gravitational* potential (and the linear connection is also related to gravitational properties), the three axioms (B.4.1) of electrodynamics are not contaminated by gravitational properties, in contrast to what happens in the usual textbook approach to electrodynamics. A curved metric or a non-flat linear connection do not affect (B.4.1), since these geometric objects don't enter these axioms. Up to now, we could do without a metric and without a connection. Yet we do have the basic Maxwellian structure already at our disposal. Consequently, the structure of electrodynamics that emerged so far has nothing to do with Poincaré or Lorentz covariance. The transformations involved are diffeomorphisms and frame transformations alone.

If one desires to generalize special relativity to general relativity theory or to the Einstein-Cartan theory of gravity (a viable alternative to Einstein's theory formulated in a non-Riemannian spacetime, see again Part C), then the Maxwellian structure in (B.4.1) is untouched by it. As long as a 4-dimensional connected, Hausdorff, paracompact and oriented differentiable manifold is used as spacetime, the axioms in (B.4.1) stay covariant and remain the same. In particular, arbitrary frames, holonomic and anholonomic ones, can be used for the evaluation of (B.4.1).

According to (B.4.1), the more specialized differential version of electrodynamics (skipping the boundary conditions) reads as follows:

$$\begin{array}{lll} dJ = 0\,, & f_\alpha = (e_\alpha \rfloor F)\wedge J\,, & dF = 0 \\ J = dH\,, & & F = dA\,. \end{array} \qquad \text{(B.4.2)}$$

maxeqns

This is the structure that we defined by means of our axioms so far. But, in fact, we know a bit more: Because of the existence of conductors and superconductors, we can *measure* the excitation H. Thus, even if H emerges as a kind of a potential for the electric current, it is more than that: It is measurable. This is in clear contrast to the potential A that is not measurable. Thus we have to take the gauge invariance under the substitution

$$A' = A + \chi\,, \quad \text{with} \quad d\chi = 0\,, \qquad \text{(B.4.3)}$$

gauge

very seriously.

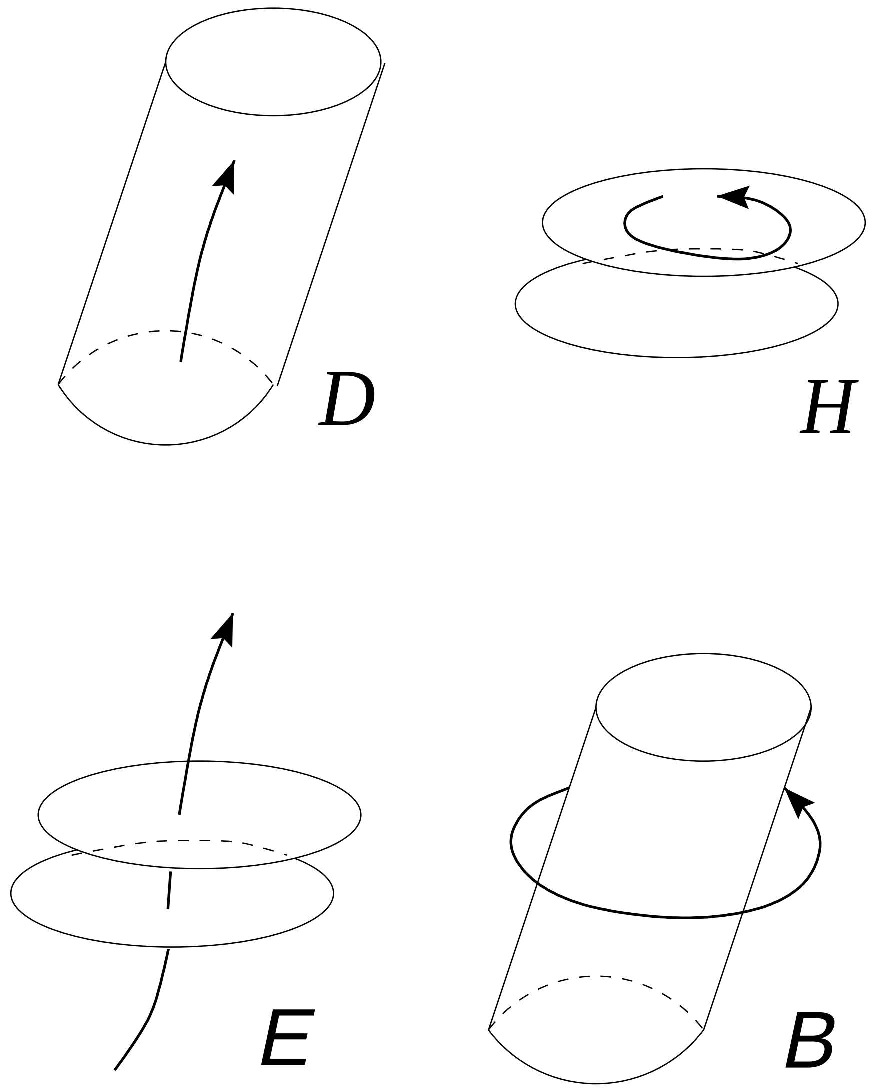

Figure B.4.1: Faraday-Schouten pictograms of the electromagnetic field in 3-dimensional space. The images of 1-forms are represented by two neighboring surfaces. The nearer the surfaces, the stronger the 1-form is. The 2-forms are pictured as flux tubes. The thinner the tubes are, the stronger the flow is. The difference between a twisted and an untwisted form accounts for the two different types of 1- and 2-forms, respectively.

The system (B.4.2) can straightforwardly be translated into the Excalc language: We denote the electromagnetic potential A by pot1. The '1' we wrote in order to remember better that the potential is a 1-form. The field strength F is written as farad2, i.e., as the Faraday 2-form. The excitation H will be named excit2, the left hand sides of the homogeneous and the inhomogeneous Maxwell equations be called maxhom3 and maxinh3, respectively. Then we need the electric current density curr3 and the left hand side of the continuity equation cont4. For the first axion, we have

```
pform cont4=4, {curr3,maxinh3}=3, excit2=2$

cont4       := d curr3;
maxinh3     := d excit2;
```

and for the second and third axiom,

```
pform force4(a)=4, maxhom3=3, farad2=2, pot1=1$
                              % has to be preceded by
frame e$                      % a coframe  statement
farad2      :=  d pot1;
force4(-a)  := (e(-a) _|farad2)^curr3;
maxhom3     :=  d farad2;
```

These program bits and pieces, which look almost trivial, will be integrated into a complete and executable Maxwell sample program after we will have learned about the energy-momentum distribution of the electromagnetic field and about its action.

The physical interpretation of the equations (B.4.2) can be found via the (1+3)-decomposition that we had derived earlier as

$$J = -j \wedge d\sigma + \rho\,, \qquad \text{(B.4.4)} \quad \text{sumj}$$

$$H = -\mathcal{H} \wedge d\sigma + \mathcal{D}\,, \qquad \text{(B.4.5)} \quad \text{sumh}$$

$$F = E \wedge d\sigma + B\,, \qquad \text{(B.4.6)} \quad \text{sumf}$$

$$A = \varphi\, d\sigma + \mathcal{A}\,, \qquad \text{(B.4.7)} \quad \text{suma}$$

see (B.1.33), (B.1.36), (B.2.7), and (B.3.7), respectively.

We first concentrate on equations which contain only measurable quantities, namely the Maxwell equations. For their (1+3)-decomposition we found, see (B.1.40,B.1.41) and (B.3.13,B.3.14),

$$dH = J \quad \begin{cases} \underline{d}\,\mathcal{D} \;= \rho & \text{(1 constraint eq.)}\,, \\ \;\dot{\mathcal{D}} \;= \underline{d}\,\mathcal{H} - j & \text{(3 time evol. eqs.)}\,, \end{cases} \tag{B.4.8}$$

evol1

$$dF = 0 \quad \begin{cases} \underline{d}\,B \;= 0 & \text{(1 constraint eq.)}\,, \\ \;\dot{B} \;= -\underline{d}\,E & \text{(3 time evol. eqs.)}\,. \end{cases} \tag{B.4.9}$$

evol2

Accordingly, we have $2 \times 3 = 6$ time evolution equations for the $2 \times 6 = 12$ variables $(\mathcal{D}, B, \mathcal{H}, E)$ of the electromagnetic field. Thus the Maxwellian structure in (B.4.2) is under-determined. We need, in addition, an electromagnetic spacetime relation that expresses the excitation $H = (\mathcal{H}, \mathcal{D})$ in terms of the field strength $F = (E, B)$, i.e., $H = H[F]$. For classical electrodynamics, this functional becomes the Maxwell-Lorentz spacetime relation that we will discuss in Part D.

Whereas, on the unquantized level, the Maxwellian structure in (B.4.1) or (B.4.2) is believed to be of universal validity, the spacetime relation is more of an ad hoc nature and amenable to corrections. The 'vacuum' can have different spacetime relations depending on whether we take it with or without vacuum polarization. We will come back to this question in Chapter E.2.

Tables (put inside the **cover of the book***)*

Table I. The electromagnetic field and its source

Field	name	math. object	independent components	related to	reflec-tion	absolute dimension
ρ	electric charge	twisted 3-form	ρ_{123}	volume	$-\rho$	$q =$ electric charge
j	electric current	twisted 2-form	j_{23}, j_{31}, j_{12}	area	$-j$	q/t
$\mathcal{D}$	electric excitation	twisted 2-form	$\mathcal{D}_{23}, \mathcal{D}_{31}, \mathcal{D}_{12}$	area	$-\mathcal{D}$	q
$\mathcal{H}$	magnetic excitation	twisted 1-form	$\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$	line	$-\mathcal{H}$	q/t
E	electric field strength	untwisted 1-form	E_1, E_2, E_3	line	E	Φ_0/t
B	magnetic field strength	untwisted 2-form	B_{23}, B_{31}, B_{12}	area	B	$\Phi_0 =$ mag-netic flux

Table II. SI-units of the electromagnetic field and its source ($C =$ coulomb, $A =$ ampere, $Wb =$ weber, $V =$ volt, $T =$ tesla; $m =$ meter, $s =$ second. The units oersted and gauss are phased out and do not exist any longer in SI)

Field	SI-unit of field	SI-unit of components of field
ρ	C	C/m^3
j	$A = C/s$	$A/m^2 = C/(sm^2)$
$\mathcal{D}$	C	C/m^2
$\mathcal{H}$	$A = C/s$	$A/m = C/(sm)$ ($\rightarrow$ oersted)
E	$Wb/s = V$	$V/m = Wb/(sm)$
B	Wb	$Wb/m^2 = T$ ($\rightarrow$ gauss)

B.4.2 ${}^{\otimes}$Jump conditions for electromagnetic excitation and field strength

The equations (B.3.34) and (B.3.38) represent the so-called jump (or continuity) conditions for the components of the electromagnetic excitation. In this section we will give a more convenient fomulation of these conditions. Namely, let us consider on a 3-dimensional slice h_σ of spacetime, see Fig.B.1.3, an arbitrary 2-dimensional surface S, the points of which are defined by the parametric equations

$$x^a = x^a(\xi^{\dot{1}}, \xi^{\dot{2}})\,, \quad a = 1, 2, 3\,. \tag{B.4.10}$$

Here $\xi^A = (\xi^{\dot{1}}, \xi^{\dot{2}})$ are the two parameters specifying the position on S. We will denote the corresponding indices $A, B, ... = \dot{1}, \dot{2}$ by a dot in order to distinguish them from the other indices. We assume that this surface is not moving, i.e. its form and position are the same in every $\sigma =$ const hypersurface. We introduce the 1-form density ν *normal* to the surface S,

$$\nu := \left(\frac{\partial x^2}{\partial \xi^{\dot{1}}}\frac{\partial x^3}{\partial \xi^{\dot{2}}} - \frac{\partial x^2}{\partial \xi^{\dot{2}}}\frac{\partial x^3}{\partial \xi^{\dot{1}}}\right) dx^1 + \left(\frac{\partial x^3}{\partial \xi^{\dot{1}}}\frac{\partial x^1}{\partial \xi^{\dot{2}}} - \frac{\partial x^3}{\partial \xi^{\dot{2}}}\frac{\partial x^1}{\partial \xi^{\dot{1}}}\right) dx^2 + \left(\frac{\partial x^1}{\partial \xi^{\dot{1}}}\frac{\partial x^2}{\partial \xi^{\dot{2}}} - \frac{\partial x^1}{\partial \xi^{\dot{2}}}\frac{\partial x^2}{\partial \xi^{\dot{1}}}\right) dx^3, \tag{B.4.11}$$

nuS

and the two vectors *tangential* to S,

$$\tau_A := \frac{\partial x^a}{\partial \xi^A}\,\partial_a\,, \qquad A = \dot{1}, \dot{2}. \tag{B.4.12}$$

tauS

Accordingly, we have the two conditions $\tau_A \rfloor \nu = 0$. The surface S divides the whole slice h_σ into two halves. We denote them by the subscripts $_{(1)}$ and $_{(2)}$, respectively.

Then, by repeating the limiting process for the integrals near S of above, we find, instead of (B.3.34) and (B.3.38), the jump conditions

$$\left(\mathcal{D}_{(2)} - \mathcal{D}_{(1)} - \widetilde{\rho}\right)\Big|_{\mathrm{S}} \wedge \nu = 0, \tag{B.4.13}$$

DonSnu

$$\tau_A \rfloor \left(\mathcal{H}_{(2)} - \mathcal{H}_{(1)} - \widetilde{j}\right)\Big|_{\mathrm{S}} = 0. \tag{B.4.14}$$

HonStau

Here, $\mathcal{D}_{(1)}$ and $\mathcal{H}_{(1)}$ are the excitation forms in the first half and $\mathcal{D}_{(2)}$ and $\mathcal{H}_{(2)}$ in the second half of the 3D space h_σ.

One can immediately verify that the analysis in Chap.B.3.4 of the operational determination of the electromagnetic excitation was carried out for the special case when the surface S was defined by the equations $x^1 = \xi^{\hat{1}}$, $x^2 = \xi^{\hat{2}}$, $x^3 = 0$ (then $\nu = dx^3$, and $\tau_A = \partial_A$, $A = 1, 2$). It should be noted though that the formulas (B.3.34) and (B.3.38) are not less general than (B.4.13) and (B.4.14) since the local coordinates can always be chosen in such a way that (B.4.13), (B.4.14) reduce to (B.3.34), (B.3.38). However, in practical applications, the use of (B.4.13), (B.4.14) turns out to be more convenient.

In an analogous way, one can derive the jump conditions for the components of the electromagnetic field strength. Starting from the homogeneous Maxwell equations (B.3.13) and (B.3.14) and considering their integral form near S, we obtain

$$\left(B_{(2)} - B_{(1)}\right)\Big|_{\rm S} \wedge \nu = 0, \qquad \text{(B.4.15)}$$ BonSnu

$$\tau_A \rfloor \left(E_{(2)} - E_{(1)}\right)\Big|_{\rm S} = 0. \qquad \text{(B.4.16)}$$ EonStau

Similar as above, $B_{(1)}$ and $E_{(1)}$ are the field strength forms in the first half and $E_{(2)}$ and $E_{(2)}$ in the second half of the 3D space h_σ.

The homogeneous Maxwell equation does not contain charge and current sources. Thus, equations (B.4.15), (B.4.16) describe the continuity of the tangential piece of the magnetic field and of the tangential part of the electric field across the boundary between the two domains of space. In (B.4.13) and (B.4.14), the components of the electromagnetic excitation are not continuous in general, with the surface charge and current densities $\widetilde{\rho}, \widetilde{j}$ defining the corresponding discontinuities.

B.4.3 Arbitrary local non-inertial frame: Maxwell's equations in components

In (B.4.8) and (B.4.9), we displayed the Maxwell equations in terms of geometrical objects in a coordinate and frame invariant way. Sometimes it is necessary, however, to introduce locally arbitrary (co-)frames of reference that are non-inertial in general. Then the components of the electromagnetic field with respect to a coframe ϑ^α, the physical components emerge and the Maxwell equations can be expressed in terms of these physical components.

Let the current, the excitation, and the field strength be decomposed according to

$$J = \frac{1}{3!} J_{\alpha\beta\gamma}\, \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma\,, \qquad \text{(B.4.17)}$$ curcomp'

$$H = \frac{1}{2} H_{\alpha\beta}\, \vartheta^\alpha \wedge \vartheta^\beta\,, \qquad F = \frac{1}{2} F_{\alpha\beta}\, \vartheta^\alpha \wedge \vartheta^\beta\,, \qquad \text{(B.4.18)}$$ maxcomp'

respectively. We substitute these expressions into the Maxwell equations. Then the coframe needs to be differentiated. As a shorthand notation, we introduce the anholonomicity 2-form $C^\alpha := d\vartheta^\alpha = \frac{1}{2} C_{\mu\nu}{}^\alpha\, \vartheta^\mu \wedge \vartheta^\nu$, see (A.2.35). Then we straightforwardly find:

$$\partial_{[\alpha} H_{\beta\gamma]} - C_{[\alpha\beta}{}^\delta H_{\gamma]\delta} = \frac{1}{3}\, J_{\alpha\beta\gamma}\,, \quad \partial_{[\alpha} F_{\beta\gamma]} - C_{[\alpha\beta}{}^\delta F_{\gamma]\delta} = 0\,. \qquad \text{(B.4.19)}$$ maxanh

If we use the (metric-free) Levi-Civita tensor density $\epsilon^{\alpha\beta\gamma\delta}$,

$$\mathcal{H}^{\alpha\beta} := \frac{1}{2!}\, \epsilon^{\alpha\beta\gamma\delta}\, H_{\gamma\delta}\,, \qquad \mathcal{J}^\alpha := \frac{1}{3!}\, \epsilon^{\alpha\beta\gamma\delta}\, J_{\beta\gamma\delta}\,, \qquad \text{(B.4.20)}$$ maxlevi

then the excitation and the current are represented as densities. Accordingly, Maxwell's equations (B.4.2) in components read alternatively

$$\partial_\beta \mathcal{H}^{\alpha\beta} + C_{\beta\gamma}{}^\alpha\, \mathcal{H}^{\beta\gamma} = \mathcal{J}^\alpha\,, \quad \partial_{[\alpha} F_{\beta\gamma]} - C_{[\alpha\beta}{}^\delta F_{\gamma]\delta} = 0\,. \qquad \text{(B.4.21)}$$ maxcomp

The terms with the C's emerge in *non-inertial* frames, i.e., they represent so-called inertial terms. If we restrict ourselves to natural (or coordinate) frames, then $C = 0$, and Maxwell's equations display their conventional form.[1]

This representation of electrodynamics can be used in special or in general relativity. If one desires to employ a laboratory frame of reference, then this is the way to do it: The object of anholonomicity in the lab frame has to be calculated. By substituting it into (B.4.19) or (B.4.21), we find the Maxwell equations in terms of the components $F_{\alpha\beta}$ etc. of the electromagnetic field quantities with respect to the lab frame – and these are the quantities one observes in the laboratory. Therefore the $F_{\alpha\beta}$ etc. are called *physical components* of F etc. As soon as one starts from (B.4.2), the derivation of the sets (B.4.19) or (B.4.21) is an elementary exercise. Many discussions of the Maxwell equations within special relativity in non-inertial frames could be appreciable shortened by using this formalism.

B.4.4 $^{\otimes}$Electrodynamics in flatland: 2-dimensional electron gas and quantum Hall effect

Our formulation of electrodynamics can be generalized straightforwardly to arbitrary dimensions n. If we assume again the charge conservation law as first axiom, then the rank of the electric current must be $n-1$. The force density in mechanics, in accordance with its definition $\sim \partial L/\partial x^i$ within the Lagrange formalism, should remain a covector-valued n-form. Hence we keep the second axiom in its original form. Accordingly, the field strength F is again a 2-form:

$$\oint_{C_{n-1}} J = 0\,, \qquad f_\alpha = (e_\alpha \rfloor F)\wedge J\,, \qquad \oint_{C_2} F = 0\,. \qquad \text{(B.4.22)}$$ 3axiomsn

This may seem like an academic exercise. However, at least for $n = 3$, there exists an application: Since the middle of the 1960's,

[1] For formulating electrodynamics in accelerated systems in terms of tensor analysis, see J. Van Bladel [37]. New experiments in rotating frames (with ring lasers) can be found in Stedman [35].

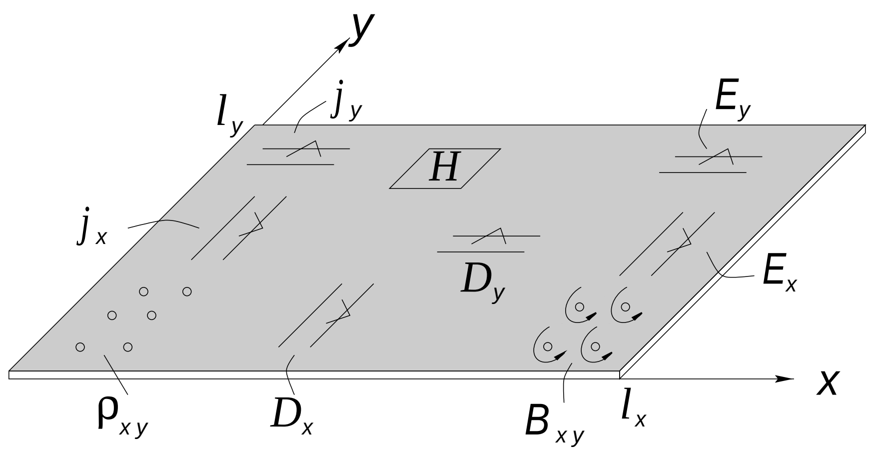

Figure B.4.2: The arsenal of electromagnetic quantities in flatland: The sources are the charge density ρ_{xy} and the current density (j_x, j_y), see (B.4.28). The excitations $(\mathcal{D}_x, \mathcal{D}_y)$ and $\mathcal{H}$ (twisted scalar) are somewhat unusual, see (B.4.29). The double plates for measuring $\mathcal{D}$, e.g., become double wires. The magnetic field has only one independent component B_{xy}, whereas the electric field has two, namely (E_x, E_y), see (B.4.30).

experimentalists were able to create a *2-dimensional electron gas* (2DEG) in suitable transistors at sufficiently low temperatures and to position the 2DEG in a strong external trasversal magnetic field. Under such circumstances, the electrons can only move in a plane transverse to B and one space dimension can be suppressed.

Electrodynamics in $1+2$ dimensions

In electrodynamics with 1 time and 2 space dimensions, we have from the first and the third axiom,

$$d \overset{(2)}{J} = 0\,, \qquad \overset{(2)}{J} = d \overset{(1)}{H}\,, \tag{B.4.23}$$ first3

and

$$d \overset{(2)}{F} = 0\,, \qquad \overset{(2)}{F} = d \overset{(1)}{A}\,, \tag{B.4.24}$$ third3

respectively, where we indicated the rank of the forms explicitly for better transparency. Here, the remarkable feature is that field strength F and current J carry the same rank; this is only possible for $n = 3$ spacetime dimensions. Moreover, the current J, the excitation H, and the field strength F all have the same number of independent components, namely 3.

Now we $(1+2)$-decompose the current and the electromagnetic field:

$$\text{twisted 2-form:} \qquad \overset{(2)}{J} = -\,j \wedge d\sigma + \rho\,, \qquad \text{(B.4.25)} \quad \texttt{zerj3}$$

$$\text{twisted 1-form:} \qquad \overset{(1)}{H} = -\mathcal{H}\; d\sigma + \mathcal{D}\,, \qquad \text{(B.4.26)} \quad \texttt{zerh3}$$

$$\text{untwisted 2-form:} \qquad \overset{(2)}{F} = \quad E \wedge d\sigma + B\,. \qquad \text{(B.4.27)} \quad \texttt{zerf3}$$

Accordingly, in the space of the 2DEG, we have

$$j = \quad j_1\, dx^1 + j_2\, dx^2\,, \qquad \rho = \rho_{12}\, dx^1 \wedge dx^2\,, \qquad \text{(B.4.28)} \quad \texttt{flatj}$$

$$\mathcal{H}\,, \qquad \mathcal{D} = \mathcal{D}_1\, dx^1 + \mathcal{D}_2\, dx^2\,, \qquad \text{(B.4.29)} \quad \texttt{flath}$$

$$E = E_1\, dx^1 + E_2\, dx^2\,, \qquad B = B_{12}\, dx^1 \wedge dx^2\,. \qquad \text{(B.4.30)} \quad \texttt{flate}$$

We recognize the rather degenerate nature of such a system. The magnetic field B, for example, has only 1 independent component B_{12}. Such a configuration is depicted in Fig. B.4.2.

Charge conservation (B.4.23) in decomposed form and in components reads,

$$\underline{d}\, j + \dot{\rho} = 0\,, \qquad \partial_1 j_2 - \partial_2 j_1 + \dot{\rho}_{12} = 0\,. \qquad \text{(B.4.31)} \quad \texttt{charge3}$$

The $(1+2)$-decomposed Maxwell equations look exactly as in (B.4.8) and (B.4.9). We will also express them in components. We find:

$$\underline{d}\,\mathcal{D} \;= \rho\,, \qquad \partial_1 \mathcal{D}_2 - \partial_2 \mathcal{D}_1 = \rho_{12}\,, \qquad \text{(B.4.32)} \quad \texttt{3max1}$$

$$\underline{d}\,\mathcal{H} - \dot{\mathcal{D}} \;= j\,, \qquad \partial_1 \mathcal{H} - \dot{\mathcal{D}}_1 = j_1\,, \qquad \text{(B.4.33)} \quad \texttt{3max2}$$

$$\partial_2 \mathcal{H} - \dot{\mathcal{D}}_2 = j_2\,, \qquad \text{(B.4.34)} \quad \texttt{3max3}$$

and

$$\underline{d}\, E + \dot{B} = 0\,, \qquad \partial_1 E_2 - \partial_2 E_1 + \dot{B}_{12} = 0\,, \qquad \text{(B.4.35)} \quad \texttt{3max4}$$

$$\underline{d}\, B \;\equiv 0\,. \qquad \text{(B.4.36)} \quad \texttt{3max5}$$

We will assume infinite extension of flatland. If that cannot be assumed as a valid approximation, one has to allow for line currents at the boundary of flatland ("edge currents") in order to fulfill the Maxwell equations.

In our formulation the Maxwell equations don't depend on the metric. Thus, instead of the Euclidean plane, as in Fig. B.4.2, we could have drawn an *arbitrary* 2-dimensional manifold, a surface of a cylinder or of a sphere, e.g.. The Maxwell equations (B.4.32) to (B.4.36) would still be valid.

Before we can apply this formalism to the quantum Hall effect (QHE), we first remind ourselves of the classical Hall effect (of 1879).

Hall effect[2]

We connect the two yz-faces of a (semi-)conducting plate of volume $l_x \times l_y \times l_z$ with a battery, see Fig. B.4.3. A current I will flow and in the plate the current density j_x. Transverse to the current, between the contacts P and Q, there exists no voltage. However, if we apply a constant magnetic field B along the z-axis, then the current j is deflected by the Lorentz force and the Hall voltage $U_{\rm H}$ occurs which, according to experiment, turns out to be

$$U_{\rm H} = R_{\rm H}\, I = A_{\rm H}\, \frac{BI}{l_z}\,, \qquad \text{with} \qquad [A_{\rm H}] = \frac{l^3}{q}\,. \qquad \text{(B.4.37)}$$ Hallvoltage

$R_{\rm H}$ is called the Hall resistance and $A_{\rm H}$ the Hall constant. We divide $U_{\rm H}$ by l_y. Because of $E_y = U_{\rm H}/l_y$, we find

$$E_y = A_{\rm H}\, B_{xy}\, \frac{I_x}{l_y\, l_z} = A_{\rm H}\, B_{xy}\, j_x\,. \qquad \text{(B.4.38)}$$ Hallvoltage'

Let us stress that the classical Hall effect is a volume (or bulk) effect. It is to be described in the framework of ordinary $(1+3)$-dimensional Maxwellian electrodynamics.

[2]See Landau-Lifshitz [21] pp.96-98 or Raith [29] p.502.

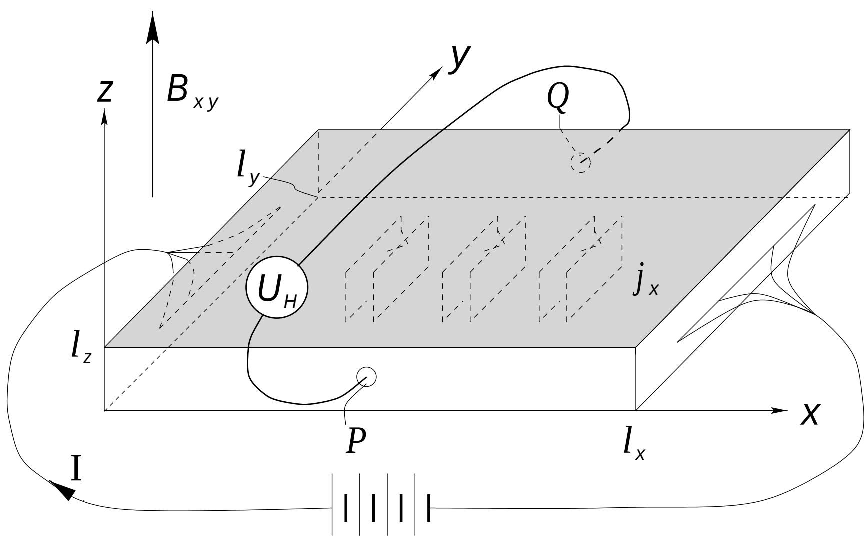

Figure B.4.3: Hall effect (schematic): The current density j in the conducting plate is affected by the external constant magnetic field B (in the figure only symbolized by one arrow) such as to create the Hall voltage U_{H}.

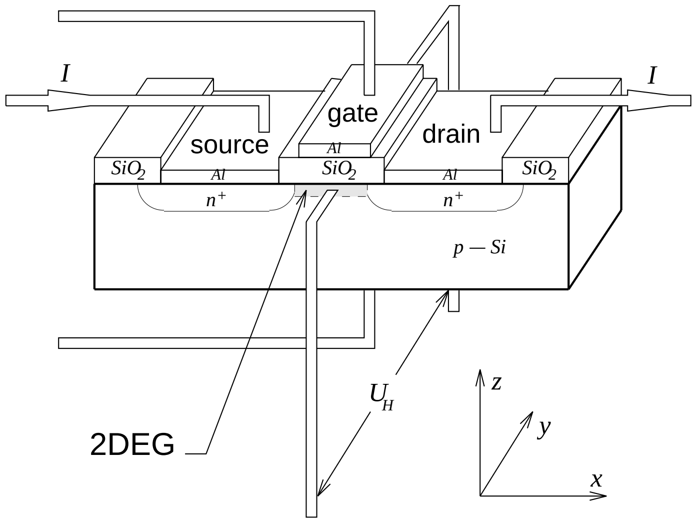

Figure B.4.4: A Mosfet with a 2-dimensional electron gas (2DEG) layer between a semiconductor (Si) and an insulator (SiO_2). Adapted from Braun [3]. In 1980, with such a transistor, von Klitzing et al. performed the original experiment on the QHE.

Quantum Hall effect[3]

A prerequisite for the discovery, in 1980, of the QHE were the advances in transistor technology. Since the 1960's one was able to assemble 2-dimensional electron gas layers in certain types of transistors, such as in a **m**etal-**o**xide-**s**emiconductor **f**ield **e**ffect **t**ransistor[4] (Mosfet), see a schematic view of a Mosfet in Fig.B.4.4. The electron layer is only about 50 nanometers thick, whereas its lateral extension may go up to the millimeter region.

In the quantum Hall regime , we have very low temperatures (between 25 mK and 500 mK) and very high magnetic fields (between 5 T and 15 T). Then the conducting electrons of the specimen, because of a (quantum mechanical) excitation gap, cannot move in the z-direction, they are confined to the xy-

[3]See, for example, von Klitzing [38], Braun [3], Chakraborty and Pietiläinen [4], Janssen et al. [17], and references given there.

[4]A fairly detailed description can be found in Raith [29] pp.579-582, e.g..

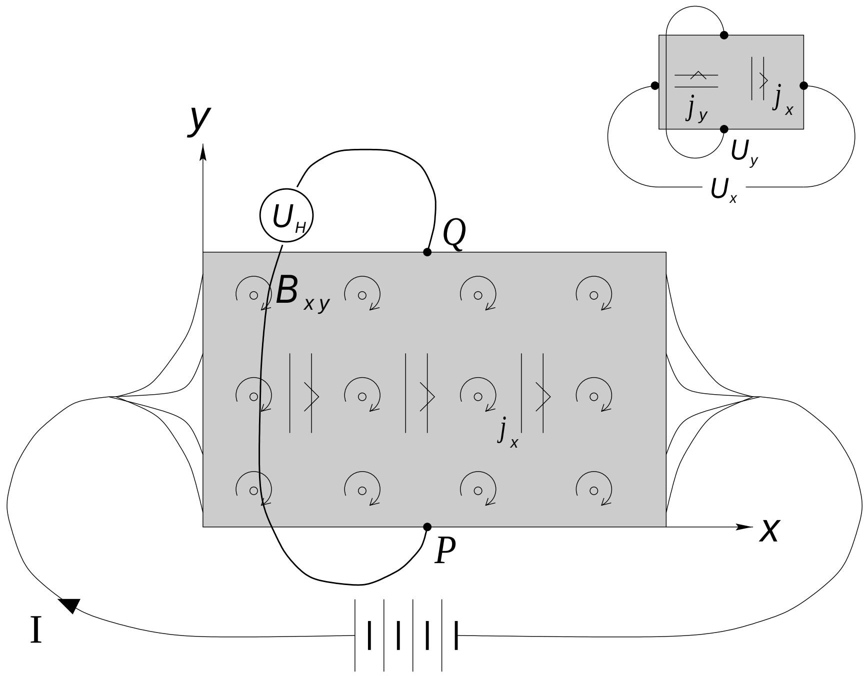

Figure B.4.5: Schematic view of a quantum Hall experiment with a 2-dimensional electron gas (2DEG). The current density j_x in the 2DEG is exposed to a strong transverse magnetic field B_{xy}. The Hall voltage U_{H} can be measured in the transverse direction to j_x between P and Q. In the inset we denoted the longitudinal voltage with U_x and the transversal one with $U_y (= U_{\mathrm{H}})$.

plane. Thus an almost ideal *2-dimensional electron gas* (2DEG) is constituted.

The Hall conductance (= 1/resistance) exhibits very well-defined plateaus at integral (and, in the fractional QHE, at rational) multiples of the fundamental conductance of $e^2/\mathrm{h} \stackrel{SI}{=} 1/(25\,812.807\,\Omega)$, where e is the elementary charge and h Planck's constant. Therefore this effect is instrumental in precision experiments for measuring, in conjunction with the Josephson effect, e and h very accurately. We will concentrate here on the *integer* QHE.

Turning to Fig.B.4.5, we consider the rectangle in the xy-plane with side lengths l_x and l_y, respectively. The quantum Hall effect is observed in such a two-dimensional system of electrons subject to a strong uniform transverse magnetic field $\vec{B}$. The configuration is similar to that of the classical Hall effect (see Fig.B.4.3), but the system is cooled down to a uniform temperature of about of 0.1 K. The *Hall resistance* R_H is defined by the ratio of the Hall voltage U_y and the electric current I_x in the x-direction: $R_\mathrm{H} = U_y/I_x$. Longituidinally, we have the ordinary dissipative Ohm resistance $R_\mathrm{L} = U_x/I_x$. For fixed values of magnetic field B and area charge density of electrons $n_e\, e$, the Hall resistance R_H is a constant. Phenomenologically, the QHE can be described by means of a *linear* tensorial Ohm-Hall law as constitutive relation. Thus, for an *isotropic* material,

$$U_x = R_\mathrm{L}\, I_x - R_\mathrm{H}\, I_y\,, \qquad \text{(B.4.39)}$$

resistance

$$U_y = R_\mathrm{H}\, I_x + R_\mathrm{L}\, I_y\,. \qquad \text{(B.4.40)}$$

If we introduce the electric field $E_x = U_x/l_x\,,\ E_y = U_y/l_y$ and the 2D current densities $j_x = I_x/l_y\,,\ j_y = I_y/l_x$, we find

$$\vec{E} = \boldsymbol{\rho}\,\vec{j} \quad \text{or} \quad \vec{j} = \boldsymbol{\sigma}\vec{E} \qquad \text{with} \qquad \boldsymbol{\rho} = \boldsymbol{\sigma}^{-1} = \begin{pmatrix} R_\mathrm{L}\frac{l_y}{l_x} & -R_\mathrm{H} \\ R_\mathrm{H} & R_\mathrm{L}\frac{l_x}{l_y} \end{pmatrix}, \qquad \text{(B.4.41)}$$

vectorQHE

where $\boldsymbol{\rho}$, the specific resistivity, and $\boldsymbol{\sigma}$, the specific conductivity, are represented by second rank tensors.

With a classical electron model for the conductivity, the Hall resistance can be calculated to be $R_\mathrm{H} = B/(n_e\, e)$, where B is the only component of the magnetic field in 2-space. The magnetic flux quantum for an electron is $\mathrm{h}/e = 2\Phi_0$, with $\Phi_0 \overset{SI}{=} 2.07 \times 10^{-15}$ Wb. With the fundamental resistance we can rewrite the Hall resistance with the dimensionless *filling factor* ν as

$$R_\mathrm{H} = \frac{B}{n_e\, e} = \frac{\mathrm{h}}{e^2}\,\nu^{-1}\,, \qquad \text{where} \qquad \nu := \frac{\frac{\mathrm{h}}{e}\, n_e}{B}\,. \qquad \text{(B.4.42)}$$

filling

Observe that ν^{-1} measures the amount of magnetic flux (in flux units) per electron.

Thus classically, if we increase the magnetic field, keeping n_e (i.e., the gate voltage) fixed, we would expect a strictly linear increasing of the Hall resistance. Surprisingly, however, we find the following (see Fig.B.4.6):

1. The Hall R_{H} has *plateaus* at rational heights. The plateaus at integer height occur with a high accuracy: $R_{\mathrm{H}} = (\mathrm{h}/e^2)/i$, for $i = 1, 2 \ldots$ (for holes we would get a negative sign).

2. When (ν, R_{H}) belongs to a plateau, the longitudinal resistance, R_{L}, *vanishes* to a good approximation.

In accordance with these results, we choose ν such that R_{L} is zero and σ_{H} constant at a plateau for one particular system, namely

$$\boldsymbol{\rho} = R_{\mathrm{H}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{or} \qquad \boldsymbol{\sigma} = \sigma_{\mathrm{H}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \qquad \text{(B.4.43)}$$

condQHE

This yields

$$\vec{j} = \sigma_{\mathrm{H}}\, \epsilon\, \vec{E} \qquad \text{with} \qquad \epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \text{(B.4.44)}$$

condQHE1

a phenomenological law valid at low frequencies and large distances. We will see in the next subsection that we can express this constitutive law in a generally covariant form. This shows that the laws governing the QHE are entirely independent of the particular geometry (metric) under consideration.

Covariant description of the phenomenology at the plateaus

For a $(1+2)$-dimensional covariant formulation[5], let us turn back to the first and the third axioms in (B.4.23) and (B.4.24), respectively. In (B.4.44), $\vec{j}$ is linearly related to $\vec{E}$. Therefore the simplest $(1+2)$-covariant ansatz is

$$\boxed{J = -\sigma_{\mathrm{H}}\, F\,,} \qquad [\sigma_{\mathrm{H}}] = q^2/h = 1/\mathit{resistance}\,. \qquad \text{(B.4.45)}$$

QHEconstit

[5] This can be found in the work of Fröhlich et al. [11, 9, 10], see also Avron [2], Richter and Seiler [33], and references given there.

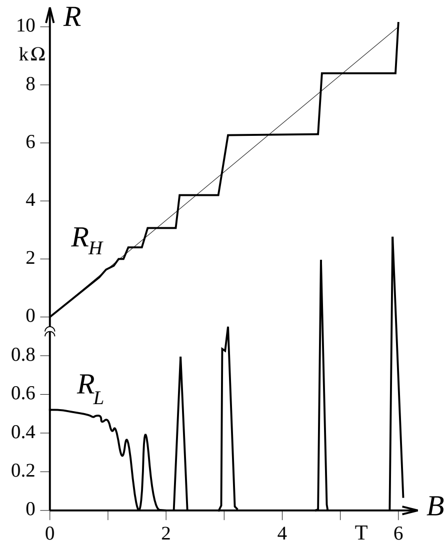

Figure B.4.6: The Hall resistance $R_{\rm H}$ at a temperature of 8 mK as a function of the magnetic field B (adapted from Ebert et al. [6]).

The minus sign is chosen since we want to recover (B.4.44). The Hall resistance $\sigma_{\rm H}$ is a twisted scalar (or pseudo-scalar). Of course, we can only hope that such a simple ansatz is valid provided the Mosfet is isotropic in the plane of the Mosfet. Because of $(B.4.23)_1$ and $(B.4.24)_1$, we find quite generally that the Hall conductance must be *constant* in space and time:

$$\sigma_{\rm H} = const. \qquad \text{(B.4.46)}$$ sigmaH

We should be aware that here happened something remarkable. We have *not* introduced a spacetime relation à la $H \sim F$, as is done in (1 + 3)-dimensional electrodynamics, see Part D. Since, for $n = 3$, H and F have still the same number of independent components, this would be possible. However, it is the ansatz $J \sim F$ that leads to a successful description of the phenomenology of the QHE. One can consider (B.4.45) as a spacetime relation for the (1 + 2)-dimensional quantum Hall regime, see Fig. B.4.7. Of course, the integer or fractional numbers of

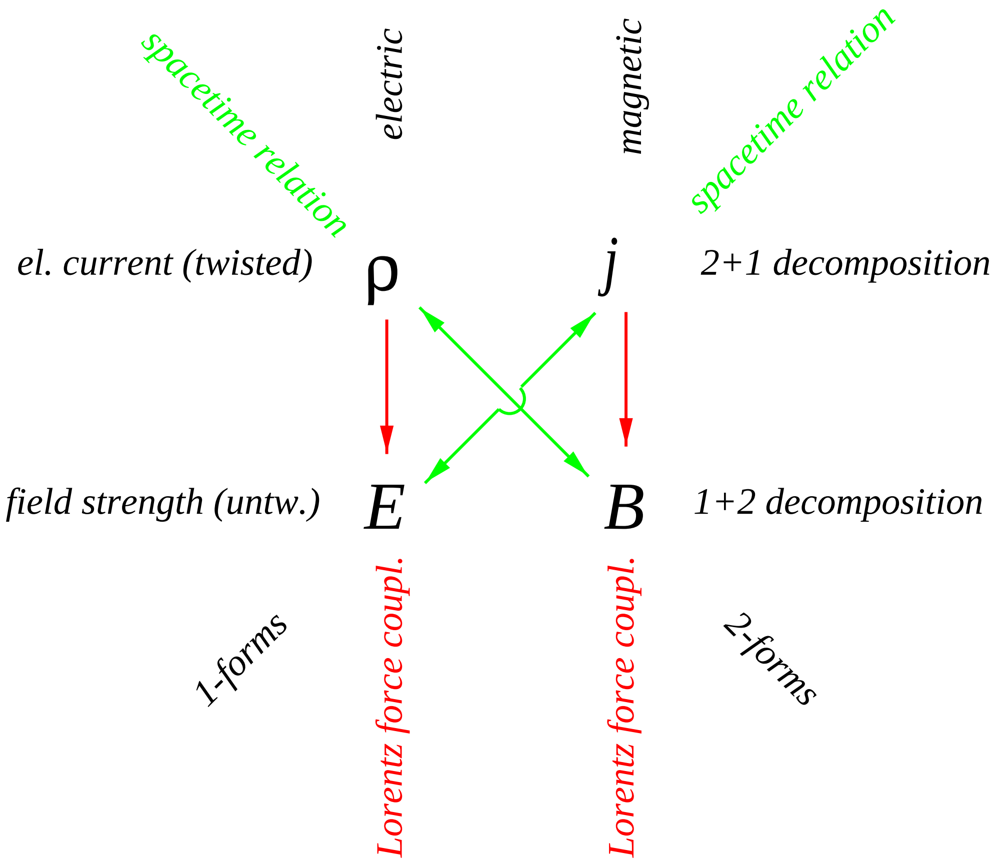

Figure B.4.7: Interrelationship between current and field strength in the Chern-Simons electrodynamics of the QHE: In *horizontal* direction, the 2-dimensional space part of a quantity is linked with its 1-dimensional time part to a 3-dimensional spacetime quantity. Lines in *vertical* direction connect a pair of quantities which contribute to the Lorentz force density. And a *diagonal* line represents a spacetime relation.

$\sigma_{\rm H}$ cannot be derived from a classical theory. Nevertheless, classical $(1+2)$-dimensional electrodynamics immediately suggests a relation of the type (B.4.45), a relation which is free of any metric. In other words, the $(1+2)$-dimensional electrodynamics of the QHE, as a classical theory, is metric-free. Eq.(B.4.45) transforms the Maxwell equations $(\text{B.4.23})_2$ and $(\text{B.4.24})_2$ to a complete system of partial differential equations which can be integrated. At the same time, is also yields an explicit relation between H and F, namely

$$dH = -\sigma_{\rm H}\, F \qquad \text{or} \qquad dH = -\sigma_{\rm H}\, dA\,. \tag{B.4.47}$$ HF3

The last equation can be integrated. We find

$$H = -\sigma_{\rm H}\, A\,, \tag{B.4.48}$$

since the potential 1-form A is only defined up to a gauge transformation anyway. The metric-free *differential* relation $(\text{B.4.47})_1$ between excitation H and field strength F is certainly not what we would have expected from classical Maxwellian electrodynamics. It represents, for $n=3$, a totally new type of (Chern-Simons) electrodynamics.

We compare (B.4.45) with (B.4.25) and (B.4.27) and find

$$j = \sigma_{\rm H}\, E \qquad \text{and} \qquad \rho = -\sigma_{\rm H}\, B\,. \tag{B.4.49}$$ QHEconstit12

In (B.4.44), the current is represented as vector. Therefore we introduce the vector density $\tilde{j} := {}^\diamond j$ by means of the diamond operator of (A.1.80), or, in components $\tilde{j}^a = \epsilon^{ab}\, j_a/2$. Then $(a,b=1,2)$,

$$\tilde{j}^a = \frac{1}{2}\,\epsilon^{ab}\, j_b = \left(\sigma_{\rm H}\,\epsilon^{ab}\right)\, E_a\,, \tag{B.4.50}$$ QHEconstit12''

that is, we recover (B.4.44) thereby verifying the ansatz (B.4.45). For the charge density $\tilde{\rho} := {}^\diamond\rho$, we find

$$\tilde{\rho} = \sigma_{\rm H}\,\epsilon^{ab}\, B_{ab} \qquad \text{with} \qquad \partial_1 \tilde{j}^1 + \partial_2 \tilde{j}^2 + \dot{\tilde{\rho}} = 0\,. \tag{B.4.51}$$

Preview: 3D Lagrangians

We will introduce Lagrangians in Sec. B.5.4. Nevertheless, let us conclude this section on electrodynamics in flatland with a few remarks on the appropriate Lagrangian for the QHE.

The Lagrangian in $(1+2)$-dimensional electrodynamics has to be a twisted 3-form with the dimension h of an action. Obviously, the first invariant of (B.2.16) qualifies,

$$I_1 = F \wedge H = -\frac{1}{\sigma_{\rm H}}\, J \wedge H = \frac{1}{\sigma_{\rm H}}\, H \wedge d\, H\,. \qquad \text{(B.4.52)}$$ lagr31

But also the Chern-Simons 3-form of (B.3.16), if multiplied by the twisted scalar $\sigma_{\rm H}$, has the right characteristics,

$$\begin{aligned} \sigma_{\rm H}\, C_A &= \sigma_{\rm H}\, A \wedge F = \sigma_{\rm H}\, A \wedge d\, A \qquad \text{(B.4.53)} \\ &= -A \wedge d\, H = d(A \wedge H) - F \wedge H = -F \wedge H + d\, K\,, \end{aligned}$$ lagr32

with the Kiehn 2-form $K = A \wedge H$. Therfore, both candidate Lagrangians are intimately linked,

$$\sigma_{\rm H}\, C_A = -I_1 + d\, K\,. \qquad \text{(B.4.54)}$$ link3

In other words, they yield the same Lagrangian since they only differ by an irrelevant exact form.

Let us define then the Lagrange 3-form

$$L_{\rm 3D} := \frac{1}{2\sigma_{\rm H}}\, H \wedge d\, H - A \wedge d\, H\,. \qquad \text{(B.4.55)}$$ lagr3

By means of its Euler-Lagrange equation,

$$d\left(\frac{\partial L}{\partial(d\, H)}\right) + \frac{\partial L}{\partial H} = 0 \qquad \text{or} \qquad \frac{1}{\sigma_{\rm H}}\, d\, H - d\, A = 0\,, \qquad \text{(B.4.56)}$$ lagr3'

we can recover the 3D spacetime relation (B.4.45) we started from. Alternatively, we can consider J as external field in the Lagrangian

$$L_{\rm 3D'} := -\frac{\sigma_{\rm H}}{2}\, A \wedge d\, A - A \wedge J\,. \qquad \text{(B.4.57)}$$ lagr3''

A similar computation as in (B.4.56) yields directly (B.4.45), q.e.d..

B.5

Electromagnetic energy-momentum current and action

B.5.1 Fourth axiom: localization of energy-momentum

Minkowski's "greatest discovery was that at any point in the electromagnetic field in vacuo *there exists a tensor of rank 2 of outstanding physical importance. ...* each component of the tensor $E_p{}^q$ has a physical interpretation, *which in every case had been discovered many years before Minkowski showed that these 16 components constitute a tensor of rank 2. The tensor* $E_p{}^q$ *is called the* energy tensor *of the electromagnetic field."*

Edmund Whittaker (1953)

Let us consider the Lorentz force density $f_\alpha = (e_\alpha \rfloor F) \wedge J$ in (B.4.2). If we want to derive the energy-momentum law for electrodynamics, we have to try to express f_α as an *exact* form. Then energy-momentum is a kind of a generalized potential for the Lorentz force density, namely $f_\alpha \sim d\,\Sigma_\alpha$. For that purpose, we start from f_α. We substitute $J = dH$ (inhomogeneous Maxwell equation) and subtract out a term with H and F exchanged and multiplied by a constant factor a:

$$f_\alpha = (e_\alpha \rfloor F) \wedge dH - a\,(e_\alpha \rfloor H) \wedge dF\,. \qquad \text{(B.5.1)}$$

inter1

Because of $dF = 0$ (homogeneous Maxwell equation), the subtracted term vanishes. The factor a will be left open for the moment. Note that we need a non-vanishing current $J \neq 0$ for our derivation to be sensible.

We partially integrate both terms in (B.5.1):

$$\begin{aligned} f_\alpha = {} & d\big[a\,F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)\big] \\ & -a\,F \wedge d(e_\alpha \rfloor H) + H \wedge d(e_\alpha \rfloor F)\,. \end{aligned} \qquad \text{(B.5.2)}$$

inter2

The first term has already the desired form. We recall the main formula for the Lie derivative of an arbitrary form Φ, namely $\mathcal{L}_{e_\alpha}\Phi = d(e_\alpha \rfloor \Phi) + e_\alpha \rfloor (d\Phi)$, see (A.2.51) This allows us to transform the second line of (B.5.2):

$$\begin{aligned} f_\alpha = {} & d\big[a\,F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)\big] \\ & -a\,F \wedge (\mathcal{L}_{e_\alpha} H) + H \wedge (\mathcal{L}_{e_\alpha} F) \\ & +a\,F \wedge e_\alpha \rfloor (dH) - H \wedge e_\alpha \rfloor (dF)\,. \end{aligned} \qquad \text{(B.5.3)}$$

inter3

The last line can be rewritten as

$$\begin{aligned} & + a\,e_\alpha \rfloor [F \wedge dH] - a\,(e_\alpha \rfloor F) \wedge dH \\ & - e_\alpha \rfloor [H \wedge dF] + (e_\alpha \rfloor H) \wedge dF\,. \end{aligned} \qquad \text{(B.5.4)}$$

inter4

As 5-forms, the expressions in the square brackets vanish. Two terms remain, and we find

$$\begin{aligned} f_\alpha = {} & d\big[a\,F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)\big] \\ & -a\,F \wedge (\mathcal{L}_{e_\alpha} H) + H \wedge (\mathcal{L}_{e_\alpha} F) \\ & -a\,(e_\alpha \rfloor F) \wedge dH + (e_\alpha \rfloor H) \wedge dF\,. \end{aligned} \qquad \text{(B.5.5)}$$

inter3a

Because of $dF = 0$, the third line adds up to $-af_\alpha$. Accordingly,

$$\begin{aligned} (1+a)\,f_\alpha = {} & d\big[a\,F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)\big] \\ & -a\,F \wedge (\mathcal{L}_{e_\alpha} H) + H \wedge (\mathcal{L}_{e_\alpha} F)\,. \end{aligned} \qquad \text{(B.5.6)}$$

inter3b

Now we have to make up our mind about the choice of the factor a. With $a = -1$, the left hand side vanishes and we find a mathematical identity. A real conservation law is only obtained when, eventually, the second line vanishes. In other words, here we need an a posteriori argument, i.e., we have to take some information from experience. For $a = 0$, the second line does not vanish. However, for $a = 1$, we can hope that the first term in the second line compensates the second term if somehow $H \sim F$. In fact, under "ordinary circumstances", to be explored below, the two terms in the second line do compensate each other for $a = 1$. Therefore we postulate this choice and find

$$f_\alpha = (e_\alpha \rfloor F) \wedge J = d\,{}^{\rm k}\Sigma_\alpha + X_\alpha\,. \qquad \text{(B.5.7)}$$ fSX

Here the *kinematic energy-momentum* 3-form of the electromagnetic field, a central result of this section, reads

$$ {}^{\rm k}\Sigma_\alpha := \frac{1}{2}\left[F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)\right] \qquad \text{(fourth axiom)}\,, \qquad \text{(B.5.8)}$$ simax

and the remaining force density 4-form turns out to be

$$X_\alpha := -\frac{1}{2}\left(F \wedge \mathcal{L}_{e_\alpha} H - H \wedge \mathcal{L}_{e_\alpha} F\right)\,. \qquad \text{(B.5.9)}$$ Xal

The absolute dimension of ${}^{\rm k}\Sigma_\alpha$ as well as of $\widehat{X}_\alpha$ is h/l.

Our derivation of (B.5.7) doesn't lead to a unique definition of ${}^{\rm k}\Sigma_\alpha$. The addition of any closed 3-form would be possible,

$${}^{\rm k}\Sigma'_\alpha := {}^{\rm k}\Sigma_\alpha + Y_\alpha\,, \qquad \text{with} \qquad dY_\alpha = 0\,, \qquad \text{(B.5.10)}$$ nonunique

such that

$$f_\alpha = d\,{}^{\rm k}\Sigma'_\alpha + X_\alpha\,. \qquad \text{(B.5.11)}$$ nonunique1

In particular, Y_α could be exact: $Y_\alpha = dZ_\alpha$. The 2-form Z_α has the same dimension as ${}^{\rm k}\Sigma_\alpha$. It seems impossible to build up Z_α exclusively in terms of the quantities e_α, H, F in an algebraic way. Therefore, $Y_\alpha = 0$ appears to be the most natural choice.

Thus, by the fourth axiom we postulate that ${}^{\mathrm{k}}\Sigma_\alpha$ *in (B.5.8) represents the energy-momentum current* that correctly localizes the energy-momentum distribution of the electromagnetic field in spacetime. We call it the *kinematic* energy-momentum current since we didn't find it by a dynamic principle, which we will not formulate before (B.5.78), but rather by means of some sort of kinematic arguments.

The current ${}^{\mathrm{k}}\Sigma_\alpha$ can also be rewritten by applying the anti-Leibniz rule for $e_\alpha\rfloor$ either in the first or the second term on the right hand side of (B.5.8). With the 4-form

$$\Lambda := -\frac{1}{2}\, F\wedge H\,, \tag{B.5.12}$$

lagr

we find

$$\begin{aligned} {}^{\mathrm{k}}\Sigma_\alpha &= \quad e_\alpha\rfloor\Lambda + F\wedge(e_\alpha\rfloor H) \\ &= -e_\alpha\rfloor\Lambda - H\wedge(e_\alpha\rfloor F)\,. \end{aligned} \tag{B.5.13}$$

simax'

B.5.2 Properties of the energy-momentum current, electric-magnetic reciprocity

${}^{\mathrm{k}}\Sigma_\alpha$ is tracefree

The energy-momentum current ${}^{\mathrm{k}}\Sigma_\alpha$ is a 3-form. We can blow it up to a 4-form according to $\vartheta^\beta\wedge{}^{\mathrm{k}}\Sigma_\alpha$. Since it still has 16 components, we haven't lost any information. If we recall that for any p-form Φ we have $\vartheta^\alpha\wedge(e_\alpha\rfloor\Phi)=p\Phi$, we immediately recognize from (B.5.8) that

$$\vartheta^\alpha\wedge{}^{\mathrm{k}}\Sigma_\alpha = 0\,, \tag{B.5.14}$$

zerotrace

which amounts to *one* equation. This property — the vanishing of the "trace" of ${}^{\mathrm{k}}\Sigma_\alpha$ — is connected with the fact that the electromagnetic field (the "photon") carries no mass and the theory is thus invariant under dilations. Why we call that the trace of the energy-momentum will become clear below, see (B.5.29).

${}^{\mathrm{k}}\Sigma_\alpha$ is electric-magnetic reciprocal

Furthermore, we can observe another property of ${}^{\mathrm{k}}\Sigma_\alpha$. It is remarkable how symmetric H and F enter (B.5.8). This was achieved by our choice of $a = 1$. The energy-momentum current is **e**lectric-**m**agnetic reciprocal, i.e., it remains invariant under the transformation

$$H \rightarrow \zeta F \,, \quad F \rightarrow -\frac{1}{\zeta} H \,, \quad \Rightarrow \quad {}^{\mathrm{k}}\Sigma_\alpha \rightarrow {}^{\mathrm{k}}\Sigma_\alpha \,, \tag{B.5.15}$$

duality1

with the twisted zero-form (pseudo-scalar function) $\zeta = \zeta(x)$ of dimension $[\zeta] = [H]/[F] = q^2/h$.

It should be stressed that in spite of ${}^{\mathrm{k}}\Sigma_\alpha$ being electric-magnetic reciprocal, Maxwell's equations are *not*,

$$dH = J \quad \rightarrow \quad dF + F \wedge d\zeta/\zeta = J/\zeta \,, \tag{B.5.16}$$

$$dF = 0 \quad \rightarrow \quad dH - H \wedge d\zeta/\zeta = 0 \,, \tag{B.5.17}$$

not even for $d\zeta = 0$, since we don't want to restrict ourselves to the free-field case with vanishing source $J = 0$.

Eq.(B.5.15) expresses a certain reciprocity between electric and magnetic effects with regard to their respective contributions to the energy-momentum current of the field. We call it electric-magnetic reciprocity.[1] That this naming is appropriate, can be seen from a (1+3)-decomposition. We recall the decompositions of H and F in (B.4.5) and (B.4.6), respectively. We substitute them in (B.5.15):

$$H \longrightarrow \zeta F \qquad \begin{cases} \mathcal{H} \longrightarrow -\zeta E \,, \\ \mathcal{D} \longrightarrow \quad \zeta B \,, \end{cases} \tag{B.5.18}$$

duality1a

$$F \longrightarrow -\frac{1}{\zeta} H \qquad \begin{cases} E \longrightarrow \quad \frac{1}{\zeta} \mathcal{H} \,, \\ B \longrightarrow -\frac{1}{\zeta} \mathcal{D} \,. \end{cases} \tag{B.5.19}$$

duality1b

[1]...following Toupin [36] even if he introduced this notion in a somewhat more restricted context. Maxwell spoke of the mutual embrace of electricity and magnetism, see Wise [39]. In the case of a *prescribed metric*, discussions of the corresponding Rainich "duality rotation" were given by Gaillard & Zumino [12] and by Mielke [26], amongst others. Note, however, that our transformation (B.5.15) is metric-free and thus of a different type.

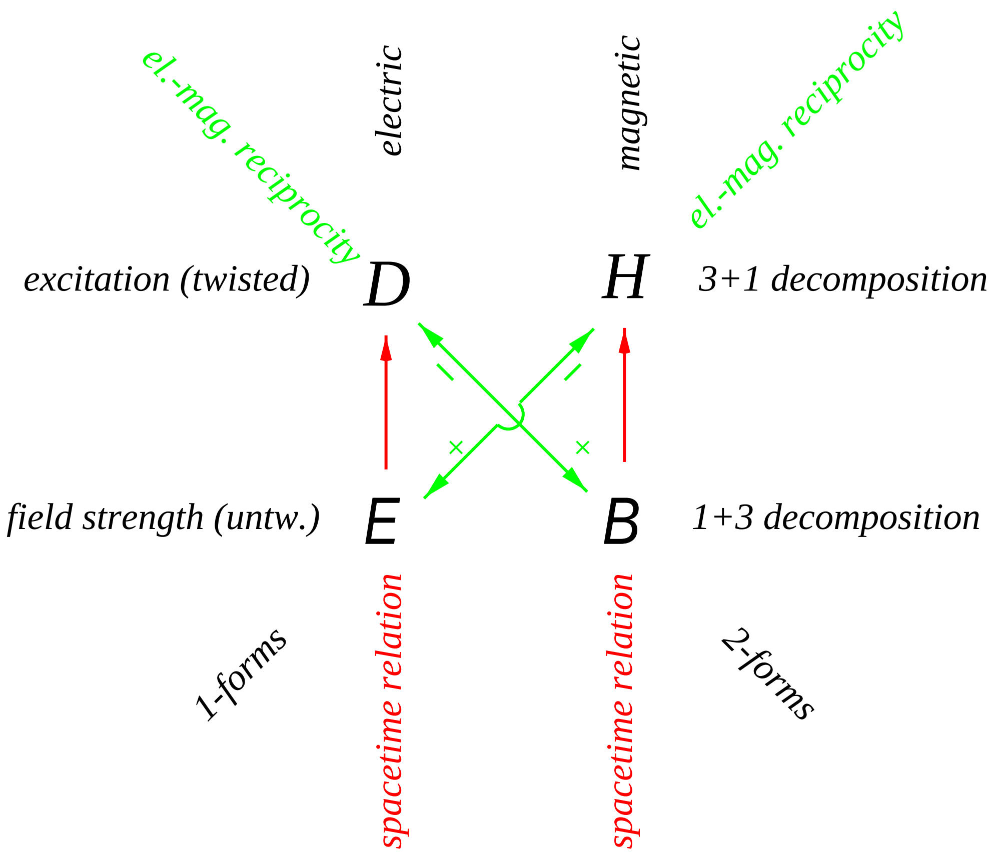

Figure B.5.1: Different aspects of the electromagnetic field: In *horizontal* direction, the 3-dimensional space part of a quantity is linked with its 1-dimensional time part to a 4-dimensional spacetime quantity. The energy-momentum current remains invariant under the exchange of those quantities that are connected by a *diagonal* line. And a *vertical* line represents a spacetime relation between quantities that are canonically related like momentum and velocity, see $(B.5.73)_1$.

Here it is clearly visible that a magnetic quantity is replaced by an electric one *and* an electric quantity by a magnetic one: *electric* $\longleftrightarrow$ *magnetic*. In this sense, we can speak of an electric-magnetic reciprocity in the expression for the energy-momentum current Σ_α. Alternatively we can say that Σ_α fulfills electric-magnetic reciprocity, it is electric-magnetic reciprocal.

Let us pause for a moment and wonder of how the notions "electric" and "magnetic" are attached to certain fields and whether there is a conventional element involved. By making experiments with a cat's skin and a rod of amber, we can "liberate" what we call *electric* charges. In 3 dimensions, they are described by the charge density ρ. Set in motion, they produce an electric current j. The electric charge is conserved (first axiom) and is linked, via the Gauss law $\underline{d}\,\mathcal{D} = \rho$, to the *electric* excitation $\mathcal{D}$.

Recurring to the Oersted experiment, it is clear that moving charges j induce magnetic effects, in accordance with the Oersted-Ampère law $\underline{d}\,\mathcal{H} - \dot{\mathcal{D}} = j$ — also a consequence of the first axiom. Hence we can unanimously attribute the term *magnetic* to the excitation $\mathcal{H}$. There is no room left for doubt about that.

The second axiom links the electric charge density ρ to the field strength E according to $(e_a \rfloor \rho) \wedge E$ and the electric current j to the field B according to $(e_a \rfloor j) \wedge B$. Consequently, also for the field strength F, there can be no other way than to label E as *electric* and B as *magnetic* field strength.

These arguments imply that the substitutions $H \longrightarrow \zeta\, F$ as well as $F \longrightarrow -H/\zeta$ both substitute an electric by a magnetic field and a magnetic by an electric one, see (B.5.18) and (B.5.19). Because of the minus sign (that is, because of $a = 1$) that we found in (B.5.15) in analyzing the electromagnetic energy-momentum current Σ_α, we can*not* speak of an equivalence of electric and magnetic fields, the expression reciprocity is much more appropriate. Fundamentally, electricity and magnetism enter into classical electrodynamics in an *a*symmetric way.

Let us try to explain the electric-magnetic reciprocity by means of a simple example. In (B.5.52) we will show that the electric energy density reads $u_{\text{el}} = \frac{1}{2}\, E \wedge \mathcal{D}$. If one wants to try to guess the corresponding expression for the magnetic energy density u_{mag}, one substitutes for an electric a corresponding magnetic quantity. However, the electric field strength is a 1-form. One cannot substitute it by the magnetic field strength B since that is a 2-form. Therefore one has to switch to the magnetic excitation according to $E \to \frac{1}{\zeta}\,\mathcal{H}$, with the 1-form $\mathcal{H}$. The function ζ is needed because of the different dimensions of E and $\mathcal{H}$ and since E is an untwisted and $\mathcal{H}$ a twisted form. Analogously, one substitutes $\mathcal{D} \to \zeta B$ thereby finding $u_{\text{mg}} = \frac{1}{2}\,\mathcal{H} \wedge B$. This is the correct result, i.e., $u = \frac{1}{2}(E \wedge \mathcal{D} + B \wedge \mathcal{H})$, and we could be happy.

Naively, one would then postulate the invariance of u under the substitution $E \to \frac{1}{\zeta}\,\mathcal{H}\,,\ \mathcal{D} \to \zeta B\,,\ B \to \frac{1}{\zeta}\,\mathcal{D}\,,\ \mathcal{H} \to \zeta E$. But, as a look at (B.4.5) and (B.4.6) will show, this cannot be implemented in a covariant way because of the minus sign in (B.4.5). One could reconsider the sign convention for $\mathcal{H}$ in (B.4.5). However, as a matter of fact, the relative sign between (B.4.5) and (B.4.6) is basically fixed by Lenz's rule (the induced electromotive force [measured in volt] is opposite in sign to the inducing field). Thus the relative minus sign is independent of conventions.

How are we going to save our rule of thumb for extracting the magnetic energy from the electric one? Well, if we turn to the substitutions (B.5.18) and (B.5.19), i.e., if we introduce two minus signs according to $E \to \frac{1}{\zeta}\,\mathcal{H}\,,\ \mathcal{D} \to \zeta B\,,\ B \to -\frac{1}{\zeta}\,\mathcal{D}\,,\ \mathcal{H} \to -\zeta E$, then u still remains invariant and we recover the covariant rule (B.5.15). In other words, the naive approach works *up to two minus signs*. Those we can supply by having insight into the covariant version of electrodynamics. Accordingly, the electric-magnetic reciprocity is the one that we knew all the time – we just have to be careful with the sign.

${}^{\text{k}}\Sigma_\alpha$ expressed in terms of the complex electromagnetic field

We can understand the electric-magnetic reciprocity transformation as acting on the column vector consisting of H and ζF:

$$\begin{pmatrix} H' \\ \zeta F' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H \\ \zeta F \end{pmatrix}. \tag{B.5.20}$$ column

In order to compactify this formula, we introduce the *complex electromagnetic field* 2-form [2]

$$U := H + i\,\zeta F \qquad \text{and} \qquad U^* = H - i\,\zeta F\,, \tag{B.5.21}$$ complex

[2]Even though we will introduce the concept of a metric only in Part C, it is necessary to point out that the complex electromagnetic *field* U, subsuming exitation and field strength (see Fig. B.5.1), should carefully be distinguished from the complex electromagnetic field *strength* introduced conventionally: $\widehat{F}^a := F^{\hat{0}a} + i\epsilon^{abc}\,F_{bc}$, with $F^{\hat{0}a} := g^{\hat{0}i} g^{aj}\,F_{ij}$. This can only be defined *after* a metric has been introduced. Similarly, for the excitation we would then have $\widehat{H}^a := -H^{\hat{0}a} + i\epsilon^{abc}\,H_{bc}$, with $H^{\hat{0}a} := g^{\hat{0}i} g^{aj}\,H_{ij}$.

with * denoting the conjugate complex. Now the electric-magnetic reciprocity (B.5.20) translates into

$$U' = -i\,U\,, \qquad U^{*\prime} = i\,U^*\,. \tag{B.5.22}$$
complex1

This corresponds, in the complex plane, where U lives, to a rotation by an angle of $-\pi/2$.

We can resolve (B.5.21) with respect to excitation and field strength:

$$H = \frac{1}{2}\,(U + U^*)\,, \qquad F = -\frac{i}{2\zeta}\,(U - U^*)\,. \tag{B.5.23}$$
complex2

We differentiate $(\text{B.5.21})_1$. Then the Maxwell equation for the complex field turns out to be

$$dU + (U^* - U)\,\frac{d\zeta}{2\zeta} = J\,. \tag{B.5.24}$$
complexmax

Clearly, if we choose a constant ζ, i.e., $d\zeta = 0$, the second term on the left hand side vanishes. The asymmetry between electric and magnetic fields finds its expression in the fact that the source term on the right hand side of (B.5.24) is a *real* quantity.

If we substitute (B.5.23) into the energy-momentum current (B.5.8), we find, after some algebra,

$$^{\rm k}\Sigma_\alpha = \frac{i}{4\zeta}\big[U^* \wedge (e_\alpha \rfloor U) - U \wedge (e_\alpha \rfloor U^*)\big]\,. \tag{B.5.25}$$
simaxcomplex

Now, according to (B.5.22), electric-magnetic reciprocity of the energy-momentum current is manifest.

If we execute successively *two* electric-magnetic reciprocity transformations, namely $U \longrightarrow U' \longrightarrow U''$, then, as can be seen from (B.5.22) or (B.5.15), we find a reflection (a rotation of $-\pi$), namely $U'' = -U$, i.e.,

$$U \rightarrow -U \qquad \text{or} \qquad (H \rightarrow -H,\; F \rightarrow -F)\,. \tag{B.5.26}$$
2duality

Only *four* electric-magnetic reciprocity transformations lead back to the identity. It should be stressed, however, that already *one* electric-magnetic reciprocity transformation leaves ${}^{\rm k}\Sigma_\alpha$ invariant.

It is now straightforward to formally extend the electric-magnetic reciprocity transformation (B.5.22) to

$$U' = e^{+i\phi}\, U\,, \qquad U^{*\prime} = e^{-i\phi}\, U^*\,, \tag{B.5.27}$$

complex1a

with $\phi = \phi(x)$ as an arbitrary "rotation" angle. The energy-momentum current ${}^{\rm k}\Sigma_\alpha$ is still invariant under this extended transformation, but in later applications only the subcase of $\phi = -\pi/2$, treated above, will be of interest.

Energy-momentum tensor density ${}^{\rm k}\mathcal{T}_\alpha{}^\beta$

Since ${}^{\rm k}\Sigma_\alpha$ is a 3-form, we can decompose it either conventionally or with respect to the basis 3-form $\hat\epsilon_\beta = e_\beta \rfloor \hat\epsilon$, with $\hat\epsilon = \vartheta^{\hat 0} \wedge \vartheta^{\hat 1} \wedge \vartheta^{\hat 2} \wedge \vartheta^{\hat 3}$, see (A.1.76):

$${}^{\rm k}\Sigma_\alpha = \frac{1}{3!}\, {}^{\rm k}\Sigma_{\lambda\mu\nu\alpha}\, \vartheta^\lambda \wedge \vartheta^\mu \wedge \vartheta^\nu =: {}^{\rm k}\mathcal{T}_\alpha{}^\beta\, \hat\epsilon_\beta\,. \tag{B.5.28}$$

emtensor

The 2nd rank tensor density of weight 1, ${}^{\rm k}\mathcal{T}_\alpha{}^\beta$, is the *Minkowski* energy tensor density. We can resolve this equation with respect to ${}^{\rm k}\mathcal{T}_\alpha{}^\beta$ by exterior multiplication with ϑ^β. We recall $\vartheta^\beta \wedge \hat\epsilon_\gamma = \delta^\beta_\gamma\, \hat\epsilon$ and find

$${}^{\rm k}\mathcal{T}_\alpha{}^\beta\, \hat\epsilon = \vartheta^\beta \wedge {}^{\rm k}\Sigma_\alpha \tag{B.5.29}$$

emtensor1

or, with the new diamond operator $^\diamond$ of (A.1.80),

$${}^{\rm k}\mathcal{T}_\alpha{}^\beta = {}^\diamond\!\left(\vartheta^\beta \wedge {}^{\rm k}\Sigma_\alpha\right) = \frac{1}{3!}\, \epsilon^{\beta\lambda\mu\nu}\, {}^{\rm k}\Sigma_{\lambda\mu\nu\alpha}\,. \tag{B.5.30}$$

emtensor1a

Thereby we recognize that $\vartheta^\alpha \wedge {}^{\rm k}\Sigma_\alpha = 0$, see (B.5.14), is equivalent to the vanishing of the trace of the energy-momentum tensor density ${}^{\rm k}\mathcal{T}_\alpha{}^\alpha = 0$. Thus ${}^{\rm k}\mathcal{T}_\alpha{}^\beta$ as well as ${}^{\rm k}\Sigma_\alpha$ have 15 independent components at this stage. Both quantities are equivalent.

If we substitute (B.5.8) into (B.5.30), then we can express the energy-momentum tensor density in the components of H and

F as follows:[3]

$$ {}^{\mathrm{k}}\mathcal{T}_\alpha{}^\beta = \frac{1}{4}\,\epsilon^{\beta\mu\rho\sigma}\left(H_{\alpha\mu}F_{\rho\sigma} - F_{\alpha\mu}H_{\rho\sigma}\right)\,. \tag{B.5.31} $$

emtensor2

${}^{\mathrm{k}}\mathcal{T}_\alpha{}^\beta$ alternatively derived by means of tensor calculus

We start with the Maxwell equations (B.4.21) in holonomic coordinates, i.e., in the natural frame $e_\alpha = \delta^i_\alpha\,\partial_i$:

$$ \partial_j\mathcal{H}^{ij} = \mathcal{J}^i\,, \qquad \partial_{[i}F_{jk]} = 0\,. \tag{B.5.32} $$

maxcomponents

Here $\mathcal{H}$ is defined according to $\mathcal{H}^{kl} = \frac{1}{2}\,\epsilon^{klmn}H_{mn}$, see $(B.4.20)_1$. We substitute the inhomogeneous Maxwell equation into the Lorentz force density and integrate partially:

$$ f_i = F_{ij}\mathcal{J}^j = F_{ij}\,\partial_k\mathcal{H}^{jk} = \partial_k\left(F_{ij}\mathcal{H}^{jk}\right) - \left(\partial_k F_{ij}\right)\mathcal{H}^{jk}\,. \tag{B.5.33} $$

lorentz

The last term can be rewritten by means of the homogeneous Maxwell equation, i.e.,

$$ -\partial_k F_{ij} = \partial_i F_{jk} + \partial_j F_{ki}\,, \tag{B.5.34} $$

aa

or

$$ f_i = \partial_k\left(F_{ij}\mathcal{H}^{jk}\right) + \left(\partial_i F_{jk} + \partial_j F_{ki}\right)\mathcal{H}^{jk}\,. \tag{B.5.35} $$

bb

Again, we integrate partially. This time the two last terms:

$$ \begin{aligned} f_i \quad &= \partial_k\left(F_{ij}\mathcal{H}^{jk}\right) + \partial_i\left(F_{jk}\mathcal{H}^{jk}\right) + \partial_j\left(F_{ki}\mathcal{H}^{jk}\right) \\ &\qquad -F_{jk}\,\partial_i\mathcal{H}^{jk} - F_{ki}\,\partial_j\mathcal{H}^{jk}\,. \end{aligned} \tag{B.5.36} $$

We collect the first three terms on the right hand side and substitute the left hand side of the inhomogeneous Maxwell equation into the last term:

$$ f_i = \partial_k\left(\delta^k_i F_{jl}\mathcal{H}^{jl} + 2F_{ij}\mathcal{H}^{jk}\right) - F_{jk}\,\partial_i\mathcal{H}^{jk} - F_{ik}\mathcal{J}^k\,. \tag{B.5.37} $$

dd

The last term represents the negative of a Lorentz force density, see (B.5.33). Thus we find

$$ f_i = \partial_k\left(\frac{1}{2}\delta^k_i F_{jl}\mathcal{H}^{jl} + F_{ij}\mathcal{H}^{jk}\right) - \frac{1}{2}F_{jl}\,\partial_i\mathcal{H}^{jl}\,. \tag{B.5.38} $$

ee

The first and the third term on the right hand side are of a related structure. We split the first term into two equal pieces and differentiate one piece:

$$ f_i = \partial_k\left(\frac{1}{4}\delta^k_i F_{jl}\mathcal{H}^{jl} + F_{ij}\mathcal{H}^{jk}\right) + \frac{1}{4}\left[\left(\partial_i F_{jl}\right)\mathcal{H}^{jl} - F_{jl}\,\partial_i\mathcal{H}^{jl}\right]\,. \tag{B.5.39} $$

ee1

Introducing the kinematic energy-momentum tensor density

$$ {}^{\mathrm{k}}\mathcal{T}_i^j = \frac{1}{4}\delta^j_i F_{kl}\mathcal{H}^{kl} - F_{ik}\mathcal{H}^{jk} \tag{B.5.40} $$

ff

and the force density

$$ \mathcal{X}_i := \frac{1}{4}\left[\left(\partial_i F_{jk}\right)\mathcal{H}^{jk} - F_{jk}\,\partial_i\mathcal{H}^{jk}\right]\,, \tag{B.5.41} $$

gg

we finally have the desired result,

$$ f_i = \partial_j\,{}^{\mathrm{k}}\mathcal{T}_i^j + \mathcal{X}_i\,, \tag{B.5.42} $$

hh

which is the component version of (B.5.7). By means of $(B.4.20)_1$, it is possible to transform (B.5.40) into (B.5.31).

[3] We leave it to the readers to prove the formula ${}^{\mathrm{k}}\mathcal{T}_\alpha{}^\gamma\,{}^{\mathrm{k}}\mathcal{T}_\gamma{}^\beta = \frac{1}{4}\,\delta^\beta_\alpha\,{}^{\mathrm{k}}\mathcal{T}^2$ which was derived first by Minkowski in 1907.

Preview: Covariant conservation law and vanishing extra force density $\widehat{X}_\alpha$

The Lorentz force density f_α in (B.5.7) and the energy-momentum current ${}^{\rm k}\Sigma_\alpha$ in (B.5.8) are covariant with respect to frame and coordinate transformations. Nevertheless, each of the two terms on the right hand side of (B.5.7), namely $d\,{}^{\rm k}\Sigma_\alpha$ or X_α, are *not* covariant by themselves. What can we do?

For the first three axioms of electrodynamics, the spacetime arena is only required to be a (1+3)-decomposable 4-dimensional manifold. We cannot be as economical as this in general. Ordinarily a *linear connection* $\Gamma_\alpha{}^\beta$ on that manifold is needed. The linear connection will be the guiding field that transports a vector, e.g., from one point of spacetime to a neighboring one.

The connection will only be introduced in Part C. There, the covariant exterior differential is defined as $D = d + \Gamma_\alpha{}^\beta \rho(L_\beta{}^\alpha)$, see (C.1.64). With the help of this operator, a generally covariant expression $D\,{}^{\rm k}\Sigma_\alpha$ can be constructed. Then (B.5.7) can be rewritten as

$$f_\alpha = D\,{}^{\rm k}\Sigma_\alpha + \widehat{X}_\alpha\,, \tag{B.5.43}$$

fSXgam

with the new supplementary force density

$$\widehat{X}_\alpha = \frac{1}{2}\left(H \wedge \text{Ł}_{e_\alpha} F - F \wedge \text{Ł}_{e_\alpha} H\right)\,, \tag{B.5.44}$$

Xalgam

which contains the covariant Lie derivative $\text{Ł}_\xi = D\,\xi \rfloor + \xi \rfloor D$, see (C.1.72). Note that the energy-momentum current ${}^{\rm k}\Sigma_\alpha$ remains the same, only the force density X_α gets replaced by $\widehat{X}_\alpha$. It is remarkable, in (B.5.43) [or in (B.5.7)] the energy-momentum current can be defined even if (B.5.43), as long as $\widehat{X}_\alpha \neq 0$, doesn't represent a genuine conservation law.

Only in this subsection, we will use the linear connection and the *covariant* exterior derivative, but not in the rest of this Part B. Then we will be able to show that the fourth axiom is exactly what is needed for an appropriate and consistent derivation of the conservation law for energy-momentum. Let us then exploit, as far as possible, the arbitrary linear connection $\Gamma_\alpha{}^\beta$, introduced above. As auxiliary quantities, attached to $\Gamma_\alpha{}^\beta$, we need the *torsion* 2-form T^α and the *transposed connection* 1-form $\overset{\frown}{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta$, both to be introduced in Part C in (C.1.43) and (C.1.44), respectively.

Let us now go back to the extra force density $\widehat{X}_\alpha$ of (B.5.44). What we need is the gauge covariant Lie derivative of an arbitrary 2-form $\Psi = \Psi_{\mu\nu}\,\vartheta^\mu \wedge \vartheta^\nu/2$ in terms of its components. Using the general formula (C.2.128) we have

$$\text{Ł}_{e_\alpha}\Psi = \frac{1}{2}\left(\overset{\frown}{D}_\alpha \Psi_{\mu\nu}\right)\vartheta^\mu \wedge \vartheta^\nu\,, \tag{B.5.45}$$

where $\overset{\frown}{D}_\alpha := e_\alpha \rfloor \overset{\frown}{D}$, with $\overset{\frown}{D}$ as the exterior covariant differential with respect to the transposed connection. Thus,

$$\widehat{X}_\alpha = \frac{1}{8}\left(H_{\rho\sigma}\overset{\frown}{D}_\alpha F_{\mu\nu} - F_{\rho\sigma}\overset{\frown}{D}_\alpha H_{\mu\nu}\right)\vartheta^\rho \wedge \vartheta^\sigma \wedge \vartheta^\mu \wedge \vartheta^\nu\,, \tag{B.5.46}$$

or, since $\vartheta^\rho \wedge \vartheta^\sigma \wedge \vartheta^\mu \wedge \vartheta^\nu = \epsilon^{\rho\sigma\mu\nu}\,\hat{\epsilon}$, we find, as alternative to (B.5.44),

$$\widehat{X}_\alpha = \frac{\hat{\epsilon}}{8}\,\epsilon^{\rho\sigma\mu\nu}\left(H_{\rho\sigma}\overset{\frown}{D}_\alpha F_{\mu\nu} - F_{\rho\sigma}\overset{\frown}{D}_\alpha H_{\mu\nu}\right). \tag{B.5.47}$$ Xfinal

This is as far as we can go with an arbitrary linear connection.

Now it becomes obvious of how one could achieve the vanishing of $\widehat{X}_\alpha$. Our four axioms don't make electrodynamics a complete theory. What is missing is the *electromagnetic spacetime relation* between excitation H and field strength F. Such a fifth axiom will be introduced in Chapter D.4. The starting point for arriving at such an axiom will be the *linear* ansatz

$$\mathcal{H}^{\mu\nu} = \frac{1}{2}\,\chi^{\mu\nu\rho\sigma}F_{\rho\sigma}\,, \quad \text{with} \quad \mathcal{H}^{\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\,H_{\rho\sigma}\,. \tag{B.5.48}$$

Substituted into (B.5.47), we have

$$\widehat{X}_\alpha = -\,\frac{\hat{\epsilon}}{8}\left(\overset{\frown}{D}_\alpha \chi^{\rho\sigma\mu\nu}\right)F_{\rho\sigma}F_{\mu\nu}\,. \tag{B.5.49}$$

Thus, the extra force density $\widehat{X}_\alpha$ will vanish provided $\chi^{\rho\sigma\mu\nu}$ is covariantly constant with respect to the transposed connection of the underlying spacetime. We will come back to this discussion in Sec.E.1.4.

B.5.3 Time-space decomposition of the energy-momentum current

Another theory of electricity, which I prefer, denies action at a distance and attributes electric action to tensions and pressures in an all-pervading medium, these stresses being the same in kind with those familiar to engineers, and the medium being identical with that in which light is supposed to be propagated.

James Clerk Maxwell (1870)

If we 1+3 decompose the Lorentz force and the energy-momentum current, then we arrive at the 3-dimensional version of the energy-momentum law of electrodynamics in a rather direct way. We recall that we work with a foliation-compatible frame e_α, as specified in (B.1.32), i.e. with $e_{\hat{0}} = n$, $e_a = \partial_a$, together with the transversality condition $e_a \rfloor d\sigma = 0$.

Consider the definition (B.5.8) of ${}^{\rm k}\Sigma_\alpha$. Substitute into it the (1+3)-decompositions (B.4.5) and (B.4.6) of the excitation H and the field strength F, respectively. Then, we obtain

$$
{}^{\rm k}\Sigma_{\hat{0}} = \; u - d\sigma \wedge s\,, \qquad \text{(B.5.50)} \quad \texttt{sig0}
$$

$$
{}^{\rm k}\Sigma_a = p_a - d\sigma \wedge S_a\,, \qquad \text{(B.5.51)} \quad \texttt{siga}
$$

where we introduced the *energy* density 3-form

$$
u := \frac{1}{2}\left(E \wedge \mathcal{D} + B \wedge \mathcal{H}\right)\,, \qquad \text{(B.5.52)} \quad \texttt{maxener}
$$

the *energy flux* density (or Poynting) 2-form

$$
s := E \wedge \mathcal{H}\,, \qquad \text{(B.5.53)} \quad \texttt{poynting}
$$

the *momentum* density 3-form

$$p_a := B \wedge (e_a \rfloor \mathcal{D}) \,, \qquad \text{(B.5.54)}$$ maxmom

and the Maxwell *stress* (or momentum flux density) 2-form of the electromagnetic field

$$\begin{aligned} S_a := & \frac{1}{2} \big[(e_a \rfloor E) \wedge \mathcal{D} - (e_a \rfloor \mathcal{D}) \wedge E \\ & + (e_a \rfloor \mathcal{H}) \wedge B - (e_a \rfloor B) \wedge \mathcal{H} \big] \,. \end{aligned} \qquad \text{(B.5.55)}$$ maxstress

Accordingly, we can represent the scheme (B.5.50)-(B.5.51) in the form of a 4×4 matrix (for density we use the abbreviation d.):

$$({}^{\rm k}\Sigma_\alpha) = \begin{pmatrix} \text{energy d.} & \text{energy flux d.} \\ \text{mom. d.} & \text{mom. flux d.} \end{pmatrix} = \begin{pmatrix} u & s \\ p_a & S_a \end{pmatrix} \,. \qquad \text{(B.5.56)}$$ sigmatrix1

The entries of the first column are 3-forms and those of the second column 2-forms.

The absolute dimensions of the quantities emerging in the 4×4 matrix can be determined from their respective definitions and the decompositions (B.4.5) and (B.4.6):

$$\begin{pmatrix} [u] & [s] \\ [p_a] & [S_a] \end{pmatrix} = h \begin{pmatrix} t^{-1} & t^{-2} \\ l^{-1} & (tl)^{-1} \end{pmatrix} \,. \qquad \text{(B.5.57)}$$ sigmatrix1a

The dimensions of their respective components read (here $i, j, k = 1, 2, 3$),

$$\begin{pmatrix} [u_{ijk}] & [s_{ij}] \\ [p_{ijk\,a}] & [S_{ij\,a}] \end{pmatrix} = \frac{h}{tl^3} \begin{pmatrix} 1 & l/t \\ (l/t)^{-1} & 1 \end{pmatrix} \,. \qquad \text{(B.5.58)}$$ sigmatrix1b

This coincides with the results from mechanics. A momentum flux density, e.g., should have the dimension $pv/l^3 = mv^2/l^3 = f/l^2 = \mathit{stress}$, in agreement with $[S_{ij\,a}] = h/(tl^3) = \mathit{energy}/l^3 = \mathit{stress} \overset{\rm SI}{=}$ Pascal. Note that dimensionwise the energy flux density s_{ij} of the electromagnetic field equals its momentum density $p_{ijk\,a}$ times the *square of a velocity* $(l/t)^2$.

Transvecting the Maxwell stress S_a, "familiar to engineers", with ϑ^a, we find straightforwardly

$$\vartheta^a \wedge S_a = -u\,, \tag{B.5.59}$$
stresstrace

which is the 3-dimensional version of (B.5.14). As soon as an electromagnetic spacetime relation will be available, we can relate the energy flux density $s \wedge \vartheta^a$, which has the same number of independent components as the 2-form s, namely 3, to the momentum density p_a. In Sect. E.1.4, for the Maxwell-Lorentz spacetime relation, we will prove in this way the symmetry of the energy-momentum current, see (E.1.31).

Since the Lorentz force density is longitudinal with respect to n, i.e., $f_a = {}^{\perp}f_a$, the forms u, s, p_a, and S_a are purely longitudinal, too. Eqs.(B.5.50)-(B.5.51) provide the decomposition of the energy-momentum 3-form into its 'time' and 'space' pieces. If we apply (B.1.26) to it, we find for the exterior differentials:

$$d\,{}^{\mathrm{k}}\Sigma_{\hat{0}} = d\sigma \wedge (\dot{u} + \underline{d}\,s)\,, \tag{B.5.60}$$
dsig0

$$d\,{}^{\mathrm{k}}\Sigma_{a} = d\sigma \wedge (\dot{p}_a + \underline{d}\,S_a)\,. \tag{B.5.61}$$
dsiga

Combining all the results, we eventually obtain for the (1+3)-decomposition of (B.5.7) the balance equations for the electromagnetic field energy and momentum:

$$k_{\hat{0}} = \;\dot{u}\; + \underline{d}\,s + (X_{\hat{0}})_{\perp}\,, \tag{B.5.62}$$
k0

$$k_a = \dot{p}_a + \underline{d}\,S_a + (X_a)_{\perp}\,. \tag{B.5.63}$$
ka

Observe finally that all the formulas displayed in this section are independent of any metric and/or connection.

B.5.4 Action

Why did we postpone the discussion of the Lagrange formalism for so long even if we know that this formalism helps so much in an effective organization of field-theoretical structures? We chose to base our axiomatics on the conservation laws of charge and flux, inter alia, which are amenable to direct experimental verification. And in the second axiom we used the concept of a force from mechanics that has also the appeal of being able to be grasped directly. Accordingly, the *proximity to experiment* was one of our guiding principles in selecting the axioms.

Already via the second axiom the notion of a force density came in. We know that this concept, according to $f_i \sim \partial L/\partial x^i$, has also a place in Lagrange formalism. When we "derived" the fourth axiom by trying to express the Lorentz force density f_α as an exact form $f_\alpha \sim d\,\Sigma_\alpha$, we obviously moved already towards the Lagrange formalism. This became apparent in (B.5.12): the 4-form Λ is a possible Lagrangian. Still, we proposed the energy-momentum current without appealing to a Lagrangian. This seemed to be more secure because we could avoid all the fallacies related to a not directly observable quantity like L. We were led, practically in a unique fashion, to the fourth axiom (B.5.8).

In any case, after having formulated the integral and the differential versions of electrodynamics including its energy-momentum distribution, we have enough understanding of its inner working as to be able to reformulate it in a Lagrangian form in a very straightforward way.

As we discussed in Sec. B.4.1, for the completion of electrodynamics we need an electromagnetic spacetime relation $H = H[F]$. This could be a nonlocal and nonlinear functional in general, as we will discuss in Chapter E.2. The field variables in Maxwell's equations are H and F. Therefore, the Lagrange 4-form of the electromagnetic field should depend on both of them:

$$V = V(H[F], F) = V[F]\,. \qquad \text{(B.5.64)}$$

From a dimensional point of view, it is quite obvious what type of action we would expect for the Maxwell field. For the excitation we have $[H] = q$ and for the field strength $[F] = h\,q^{-1}$. Accordingly, $\sim \int H \wedge F$ would qualify as action. And this is, indeed, what we will find out eventually.

Maxwell's equations are first order in $H = H[F]$ and F, respectively. Since $F = dA$, the field strength F itself is first order in A. We will take A as field variable. The Euler-Lagrange equations of a variational principle with a Lagrangian of differential order m are of differential order $2m$. The field equations are assumed at most of 2nd differential order in the field variables A, ψ. Therefore the Lagrangian is of 1st order in these fields.

Consider an electrically charged matter field ψ, which, for the time being, is assumed to be a p-form. The total Lagrangian of the system, a twisted 4-form, should consist of a free field part V of the electromagnetic field and a matter part L_{mat}, the latter of which describes the matter field ψ and its coupling to A:

$$L = V + L_{\text{mat}} = V(A, dA, \Phi) + L_{\text{mat}}(A, dA, \psi, d\psi, \Phi)\,. \tag{B.5.65}$$

lmatter

Here Φ are what we can call the *structural* fields. Their presence, at this stage, is motivated by the technical reasons: in order to be able to *construct* a viable Lagrangian other than a trivial ("topological") $F \wedge F$, one should have a tool which helps to do this. The role of such a tool is played by Φ. Here we do not specify the nature of the structural fields, but we will see below that as soon as the metric (and connection) are defined on the spacetime, they will represent the fields Φ properly.

We require V to be *gauge invariant*, that is

$$\delta V = V(A + \delta A, d[A + \delta A], \Phi) - V(A, dA, \Phi) = 0\,, \tag{B.5.66}$$

vinv

where $\delta A = d\omega$ represents an infinitesimal gauge transformation of the type (B.3.9) for $\Phi \approx \omega$. We obtain

$$\delta V = d\omega \wedge \frac{\partial V}{\partial A} = 0 \qquad \Leftrightarrow \qquad \frac{\partial V}{\partial A} = 0\,, \tag{B.5.67}$$

delV

or

$$V = V(dA, \Phi) = V(F, \Phi)\,. \tag{B.5.68}$$

vf

Hence the free field Maxwell or *gauge* Lagrangian can depend on the potential A only via the field strength $F = dA$. The matter Lagrangian should also be gauge-invariant.

The action reads

$$W = \int_{\Omega_4} L\,. \tag{B.5.69}$$

action

The field equations for A are given by the stationary points of W under a variation δ of A which commutes with the exterior derivative by definition, that is, $\delta d = d\delta$, and vanishes at the boundary, i.e. $\delta A|_{\partial\Omega_4} = 0$. Varying A yields

$$\begin{aligned}
\delta_A W = \int\limits_{\Omega_4} \delta_A L &= \int\limits_{\Omega_4} \left[\delta A \wedge \frac{\partial L}{\partial A} + \delta dA \wedge \frac{\partial L}{\partial dA}\right] \\
&= \int\limits_{\Omega_4} \left[\delta A \wedge \left\{\frac{\partial L}{\partial A} - (-1)^1 d\frac{\partial L}{\partial dA}\right\} + d\left\{\delta A \wedge \frac{\partial L}{\partial dA}\right\}\right] \\
&= \int\limits_{\Omega_4} \delta A \wedge \frac{\delta L}{\delta A} + \int\limits_{\partial\Omega_4} \delta A \wedge \frac{\partial L}{\partial dA}\,, \qquad \text{(B.5.70)}
\end{aligned}$$

where the variational derivative of the 1-form A is defined according to

$$\frac{\delta L}{\delta A} := \frac{\partial L}{\partial A} + d\frac{\partial L}{\partial dA}\,. \qquad \text{(B.5.71)}$$ delL

Stationarity of W leads to the *gauge field equation*

$$\frac{\delta L}{\delta A} = 0\,. \qquad \text{(B.5.72)}$$ gaugefield

Keeping in mind the inhomogeneous Maxwell field equation in (B.4.2), we define the excitation ("field momentum") conjugated to A and the matter current by

$$H = -\frac{\partial V}{\partial dA} = -\frac{\partial V}{\partial F} \qquad \text{and} \qquad J = \frac{\delta L_{\text{mat}}}{\delta A}\,, \qquad \text{(B.5.73)}$$ fieldmom

respectively. Then we recover, indeed:

$$dH = J\,. \qquad \text{(B.5.74)}$$ maxin

The homogeneous Maxwell equation is a consequence of working with the potential A, since $F = dA$ and $dF = ddA = 0$. In (B.5.74), we were also able to arrive at the inhomogeneous equation. The excitation H and the current J are, however, only implicitly given. As we can see from (B.5.73), only an explicit form of the Lagrangians V and L_{mat} promotes H and j to more than sheer placeholders. On the other hand, it is very satisfying to recover the structure of Maxwell's theory already at such an implicit level. Eq.(B.5.73)$_1$ represents the as yet unknown *spacetime relation* of Maxwell's theory.

Let us turn to the variational of the matter field ψ. Its variational derivative reads

$$\frac{\delta L}{\delta \psi} := \frac{\partial L}{\partial \psi} - (-1)^p \, d \frac{\partial L}{\partial d\psi} \, . \qquad \text{(B.5.75)}$$ varpsi

Since V does not depend on ψ, we find for the *matter field equation* simply

$$\frac{\delta L_{\text{mat}}}{\delta \psi} = 0 \, . \qquad \text{(B.5.76)}$$ delmat

All what is left to do now in this context, is to specify the spacetime relation (B.5.73)$_1$ and thereby to transform the Maxwell and the matter Lagrangian into an explicit form.

At this stage, the structural fields Φ are considered as non-dynamical ("background"), so we do not have equations of motion for them.

B.5.5 $^{\otimes}$Coupling of the energy-momentum current to the coframe

In this section we will show that the canonical definition of the energy-momentum current (as a Noether current corresponding to spacetime translations) coincides with its *dynamic* definition as a source of the gravitational field that is represented by the coframe – and both are closely related to the kinematic current of our fourth axiom.

Let us assume that the interaction of the electromagnetic field with gravity is 'switched on'. On the Lagrangian level, it means that (B.5.68) should be replaced by the Lagrangian

$$V = V(\vartheta^\alpha, F). \qquad \text{(B.5.77)}$$ vfvta

From now on, the coframe assumes the role of the structural field Φ. In a standard way, the *dynamic* (or Hilbert) energy-momentum current for the coframe field is defined by

$$ {}^{\rm d}\Sigma_\alpha := -\frac{\delta V}{\delta \vartheta^\alpha} = -\left(\frac{\partial V}{\partial \vartheta^\alpha} + d\,\frac{\partial V}{\partial (d\vartheta^\alpha)}\right) . \qquad \text{(B.5.78)} $$ SigDyn

The last term vanishes for the Lagrangian (B.5.77) under consideration. As before, cf. (B.5.73), the electromagnetic field momentum is defined by

$$ H = -\frac{\partial V}{\partial F}. \qquad \text{(B.5.79)} $$

The crucial point is the condition of general coordinate or diffeomorphism invariance of the Lagrangian (B.5.77) of the interacting electromagnetic and coframe fields. The general variation of the Lagrangian reads

$$ \delta V = \delta\vartheta^\alpha \wedge \frac{\partial V}{\partial \vartheta^\alpha} + \delta F \wedge \frac{\partial V}{\partial F} = -\,\delta\vartheta^\alpha \wedge {}^{\rm d}\Sigma_\alpha - \delta F \wedge H\,. \qquad \text{(B.5.80)} $$ varL0

If ξ is a vector field generating an arbitrary one–parameter group $\mathcal{T}_t$ of diffeomorphisms on X, the variation in (B.5.80) is described by the Lie derivative, i.e., $\delta = \mathcal{L}_\xi = \xi \rfloor d + d\xi \rfloor$. Substituting this into the left-hand and right-hand sides of (B.5.80), we find, after some rearrangements, the identity

$$ \begin{aligned} &d\Big[(\xi \rfloor V) + (\xi \rfloor \vartheta^\alpha)\,{}^{\rm d}\Sigma_\alpha + (\xi \rfloor F)\wedge H\Big] \\ &-(\xi \rfloor \vartheta^\alpha)\, d\,{}^{\rm d}\Sigma_\alpha + (\xi \rfloor d\vartheta^\alpha)\wedge {}^{\rm d}\Sigma_\alpha - (\xi \rfloor F)\wedge dH = 0. \end{aligned} \qquad \text{(B.5.81)} $$ AB0

Since ξ is arbitrary, the first and second lines are vanishing separately. Now, the final step is to put $\xi = e_\alpha$. Then (B.5.81) yields two identities.

From the first line of (B.5.81) we find that the dynamic current ${}^{\rm d}\Sigma_\alpha$, defined in (B.5.78), can be identified with the *canonical* (or Noether) *energy-momentum* current, i.e.,

$$ {}^{\rm d}\Sigma_\alpha \equiv \Sigma_\alpha \qquad \text{with} \qquad \Sigma_\alpha := -\,e_\alpha \rfloor V - (e_\alpha \rfloor F)\wedge H\,. \qquad \text{(B.5.82)} $$ sigcan

Thus we don't need to distinguish any longer between the *dynamic* and the *canonical* energy-momentum current. This fact matches very well with the structure of the *kinematic* energy-momentum current ${}^{\mathrm{k}}\Sigma_\alpha$ of (B.5.8) or, better, of (B.5.13). We compare (B.5.82) with (B.5.13). If we choose as Lagrangian $V = -F \wedge H/2$, then ${}^{\mathrm{k}}\Sigma_\alpha = \Sigma_\alpha$ – and we can drop the k and the d from ${}^{\mathrm{k}}\Sigma_\alpha$ and ${}^{\mathrm{d}}\Sigma_\alpha$, respectively. However, this choice of the Lagrangian will only be legitimized by our fifth axiom in Chapter D.5.

The second line of (B.5.81) yields the *conservation law* of energy-momentum:

$$d\,\Sigma_\alpha = (e_\alpha \rfloor d\vartheta^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor F) \wedge J, \qquad \text{(B.5.83)}$$ conserv

where we have used the inhomogeneous Maxwell equation $dH = J$. The presence of the first term on the right-hand side guarantees the covariant character of that equation. It is easy to see that we can rewrite (B.5.83) in the equivalent form

$$\widetilde{D}\,\Sigma_\alpha = (e_\alpha \rfloor F) \wedge J. \qquad \text{(B.5.84)}$$

by making use of the Riemannian covariant exterior derivative $\widetilde{D}$ to be defined in Part C.

Our discussion can even be made more general,[4] going beyond a pure electrodynamical theory. Namely, let us consider a theory of the coframe ϑ^α coupled to a *generalized matter* field Ψ. Note that the latter may not be just one function or an exterior form, but and *arbitrary collection* of forms of all possible ranks and/or exterior forms of type ρ, i.e. $\Psi = (\overset{(0)}{\psi}{}^U, \dots, \overset{(p)}{\psi}{}^A, \dots)$ (note that the range of indices for the forms of different rank is, in general, also different, that is, e.g., U and A run over different ranges). We assume that the Lagrangian of such a theory depends very generally on frame, matter field, and its derivatives:

$$L = L(\vartheta^\alpha\,, d\vartheta^\alpha\,, \Psi, d\Psi)\,. \qquad \text{(B.5.85)}$$ Lvarpsi

Normally, the matter Lagrangian does not depend on the derivatives of the coframe, but we include $d\vartheta^\alpha$ for greater generality. [It is important to realize that the Lagrangian (B.5.85) indeed describes the most general theory: For example, the set Ψ can include not only the true matter fields, described, say, by a p-form $\overset{(p)}{\psi}{}^A$, but formally also other virtual gravitational potentials such as the metric 0-form $g_{\alpha\beta}$, the connection 1-form $\Gamma_\alpha{}^\beta$, as soon as they are defined on X and are interacting with each other].

[4] See,e.g., Ref.[14].

The basic assumption about the Lagrangian (B.5.85) is that L is a *scalar-valued twisted n-form which is invariant under the spacetime diffeomorphisms.* This simple input has amazingly general consequence.

The dynamic energy-momentum current of matter will be defined as in (B.5.78),

$$ {}^{\rm d}\Sigma_\alpha := -\frac{\delta L}{\delta\vartheta^\alpha} = -\left(\frac{\partial L}{\partial\vartheta^\alpha} + d\,\frac{\partial L}{\partial(d\vartheta^\alpha)}\right). \qquad \text{(B.5.86)} \quad \texttt{SigDyn'} $$

Similarly, the variational derivative of the generalized matter reads

$$ \frac{\delta L}{\delta\Psi} = \frac{\partial L}{\partial\Psi} - (-1)^p d\,\frac{\partial L}{\partial(d\Psi)}, \qquad \text{(B.5.87)} \quad \texttt{VDpsi} $$

where the sign factor $(-1)^p$ correlates with the relevant rank of a particular component in the the set of fields Ψ. This is a simple generalization of equation (B.5.75).

According to the Noether theorem, the conservation identities of the matter system result from the postulated invariance of L under a local symmetry group. Actually, this is only true "weakly", i.e., provided the Euler–Lagrange equation (B.5.87) for the matter fields is satisfied.

Here we consider the consequences of the invariance of L under the group diffeomorphisms on the spacetime manifold. Let ξ be a vector field generating an arbitrary one–parameter group $\mathcal{T}_t$ of diffeomorphisms on X. In order to obtain a corresponding Noether identity from the invariance of L under a one–parameter group of *local* translations $\mathcal{T}_t \subset \mathcal{T} \approx Diff(4,R)$, it is important to recall that infinitesimally the action of a one–parameter group $\mathcal{T}_t$ on X is described by the conventional Lie derivative (A.2.49) with respect to a vector field ξ. Since we work with the fields which are exterior forms of various ranks, the most appropriate formula for the Lie derivative is (A.2.51), i.e. $\mathcal{L}_\xi = \xi\rfloor d + d\xi\rfloor$. The general variation of the Lagrangian (B.5.85) reads:

$$ \delta L = \delta\vartheta^\alpha\wedge\frac{\partial L}{\partial\vartheta^\alpha} + (\delta d\vartheta^\alpha)\wedge\frac{\partial L}{\partial d\vartheta^\alpha} + \delta\Psi\wedge\frac{\partial L}{\partial\Psi} + (\delta d\Psi)\wedge\frac{\partial L}{\partial d\Psi}. \qquad \text{(B.5.88)} \quad \texttt{varL} $$

For the variations generated by a one-parameter group of the vector field ξ we have to substitute $\delta = \mathcal{L}_\xi$ in (B.5.88). This is straightforward and, after performing some 'partial integrations', we find

$$ \begin{aligned} d(\xi\rfloor L) = & d\Big[(\xi\rfloor\vartheta^\alpha)\,\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor d\vartheta^\alpha)\wedge\frac{\partial L}{\partial d\vartheta^\alpha} \\ & + (\xi\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi} + (\xi\rfloor d\Psi)\wedge\frac{\partial L}{\partial d\Psi}\Big] \\ & - (\xi\rfloor\vartheta^\alpha)d\frac{\delta L}{\delta\vartheta^\alpha} + (\xi\rfloor d\vartheta^\alpha)\wedge\frac{\delta L}{\delta\vartheta^\alpha} \\ & + (\xi\rfloor d\Psi)\wedge\frac{\delta L}{\delta\Psi} + (-1)^p(\xi\rfloor\Psi)\wedge d\frac{\delta L}{\delta\Psi}. \end{aligned} \qquad \text{(B.5.89)} \quad \texttt{varL1} $$

Now we rearrange the equation of above by collecting terms under the exterior derivative separately. Then (B.5.89) can be written as

$$ A - dB = 0, \qquad \text{(B.5.90)} \quad \texttt{A+dB} $$

where

$$ \begin{aligned} A := & -(\xi\rfloor\vartheta^\alpha)d\frac{\delta L}{\delta\vartheta^\alpha} + (\xi\rfloor d\vartheta^\alpha)\wedge\frac{\delta L}{\delta\vartheta^\alpha} \\ & + (\xi\rfloor d\Psi)\wedge\frac{\delta L}{\delta\Psi} + (-1)^p(\xi\rfloor\Psi)\wedge d\frac{\delta L}{\delta\Psi}, \end{aligned} \qquad \text{(B.5.91)} \quad \texttt{defA} $$

$$ \begin{aligned} B := & \,\xi\rfloor L - (\xi\rfloor\vartheta^\alpha)\,\frac{\partial L}{\partial\vartheta^\alpha} - (\xi\rfloor d\vartheta^\alpha)\wedge\frac{\partial L}{\partial d\vartheta^\alpha} \\ & - (\xi\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi} - (\xi\rfloor d\Psi)\wedge\frac{\partial L}{\partial d\Psi}. \end{aligned} \qquad \text{(B.5.92)} \quad \texttt{defB} $$

The functions A and B have the form

$$A = \xi^\alpha A_\alpha , \qquad B = \xi^\alpha B_\alpha . \tag{B.5.93}$$

Hence, by (B.5.90),

$$\xi^\alpha (A_\alpha - dB_\alpha) - d\xi^\alpha \wedge B_\alpha = 0, \tag{B.5.94}$$

where *both* ξ^α *and* $d\xi^\alpha$ are *pointwise arbitrary*. Hence we can conclude that B_α as well as A_α vanish:

$$A = 0, \quad \text{and} \quad B = 0 . \tag{B.5.95}$$

Since the vector field ξ is arbitrary, it is sufficient to replace it via $\xi \to e_\alpha$ by the frame field. Then, for $B = 0$, we obtain from (B.5.92)

$$\begin{aligned} {}^{\rm d}\Sigma_\alpha = &- e_\alpha \rfloor L + (e_\alpha \rfloor d\Psi) \wedge \frac{\partial L}{\partial d\Psi} + (e_\alpha \rfloor \Psi) \wedge \frac{\partial L}{\partial \Psi} \\ &- d\frac{\partial L}{\partial d\vartheta^\alpha} + (e_\alpha \rfloor d\vartheta^\beta) \wedge \frac{\partial L}{\partial d\vartheta^\beta} . \end{aligned} \tag{B.5.96}$$ dyncurr

For $A = 0$, eq.(B.5.91) yields

$$d\,{}^{\rm d}\Sigma_\alpha \equiv (e_\alpha \rfloor d\vartheta^\beta) \wedge {}^{\rm d}\Sigma_\beta + \mathcal{F}_\alpha , \tag{B.5.97}$$ 1stNoe

where

$$\mathcal{F}_\alpha := -(e_\alpha \rfloor d\Psi) \wedge \frac{\delta L}{\delta \Psi} - (-1)^p (e_\alpha \rfloor \Psi) \wedge d\frac{\delta L}{\delta \Psi} . \tag{B.5.98}$$

In fact, when the matter Lagrangian is independent of the derivatives of the coframe field, i.e., $\partial L/\partial d\vartheta^\beta = 0$, eq. (B.5.96) demonstrates the equality of the dynamic energy-momentum current, that is coupled to the coframe, with the canonical one, the Noether current of the translations.

B.5.6 Maxwell's equations and the energy-momentum current in Excalc

It is a merit of exterior calculus that electrodynamics and, in particular, Maxwell's equations can be formulated in a very succinct form. This translates into an equally concise form of the corresponding computer programs in Excalc. The goal of better understanding the structure of electrodynamics leads us, hand in hand, to a more transparent and a more effective way of computer programming.

In Excalc, as we mentioned in Sec. A.2.11, the electrodynamical quantities are evaluated with respect to the coframe that is specified in the program. If we put in a accelerating and rotating coframe ω^α, e.g., then the electromagnetic field strength F in the program will be evaluated with respect to this frame:

$F = F_{\alpha\beta}\,\omega^\alpha \wedge \omega^\beta/2$. This is, of course, exactly what we discussed in Sec. B.4.3 when we introduced arbitrary noninertial coframes.

We cautioned already our readers in Sec. A.2.11 that we need to specify a coframe together with the metric. Thus, our Maxwell sample program proper, to be displayed below, is preceded by coframe and frame commands. We pick the spherical coordinate system of Sec. A.2.11 and require the spacetime to be Minkowskian, i.e., $\psi(r) = 1$.

Afterwards we specify the electromagnetic potential $A = A_\alpha\,\vartheta^\alpha$, namely `pot1`. Since we haven't defined a specific problem so far, we leave its components `aa0`, `aa1`, `aa2`, `aa3` open for the moment. Then we put in the pieces discussed subsequent to (B.4.2). In order to relate H and F, we have to make use of the fifth axiom, only to be pinned down in eq.(D.5.7):

```
% file mustermax.exi, 2001-05-31

load_package excalc$

pform psi=0$ fdomain psi=psi(r)$       % coframe defined
coframe o(0)     =  psi            * d t,
        o(1)     = (1/psi)         * d r,
        o(2)     = r               * d theta,
        o(3)     = r * sin(theta)  * d phi
   with signature (1,-1,-1,-1)$
frame e$

psi:=1;                          % flat spacetime assumed

% start of Maxwell proper: unknown functions aa0, aa1...

pform {aa0,aa1,aa2,aa3}=0, pot1=1, {farad2,excit2}=2,
      {maxhom3,maxinh3}=3$
fdomain aa0=aa0(t,r,theta,phi),aa1=aa1(t,r,theta,phi),
        aa0=aa0(t,r,theta,phi),aa1=aa1(t,r,theta,phi)$

pot1    := aa0*o(0) + aa1*o(1) + aa2*o(2) + aa3*o(3)$
```

```
farad2  := d pot1;
maxhom3 := d farad2;
excit2  := lam * # farad2;  % spacetime relation, see
maxinh3 := d excit2;        % 5th axiom Eq.(D.5.7)

% Maxwell Lagrangian and energy-momentum current assigned

pform lmax4=4, maxenergy3(a)=3$

lmax4          := -(1/2)*farad2^excit2;
maxenergy3(-a) := e(-a) _|lmax4 + (e(-a) _|farad2)^excit2;

% Use a blank before the interior product sign!
end;
```

If this sample program is written onto a file with name *mustermax.exi* (exi stands for excalc-input, the corresponding output file has the extension .exo), then this very file can be read into a Reduce session by the command `in"mustermax.exi";`

As a trivial test, you can specify the potential of a point charge sitting at the origin of our coordinate system:

```
aa0 := -q/r;       aa1 := aa2 := aa3 := 0;
```

Determine its field strength by `farad2:=farad2;` its excitation by `excit2:=excit2;` and its energy-momentum distribution by `maxenergy3(-a):maxenergy3(-a);` you will find the results you are familiar with. And, of course, you want to convince yourself that Maxwell's equations are fulfilled by releasing the commands `maxhom3:=maxhom3;` and `maxinh3:=maxinh3;`

This sample program can be edited according to the needs one has. Prescribe a non-inertial coframe, i.e., an accelerating and rotating coframe. Then you just have to edit the coframe command and can subsequently compute the corresponding physical components of an electromagnetic quantity with respect to that frame. Applications of this program include the Reissner-Nordström and the Kerr-Newman solutions of general relativity; they represent the electromagnetic and the gravitational fields of an electrically charged mass of spherical or axial symmetry, respectively. They will be discussed in the outlook in the last part of the book.

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Part C

More mathematics

C.1

Linear connection

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"...the essential achievement of general relativity, namely to overcome 'rigid' space (ie the inertial frame), is *only indirectly* connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the 'displacement field' (Γ^l_{ik}), which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame (ie the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of 'rigid' space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric..."

A. Einstein (1955 April 04)[1]

[1]Translation by F. Gronwald, D. Hartley, and F.W. Hehl from the German original: See Preface in [6].

C.1.1 Covariant differentiation of tensor fields

The change of scalar functions f along a vector u is described by the directional derivative $\partial_u f$. The generalization of this notion from scalars f to tensors T is provided by the covariant differentiation $\nabla_u T$.

When calculating a *directional derivative* of a function $f(x)$ along a vector field u, one has to know the values of $f(x)$ at different points on the integral lines of u. With the standard definition which involves taking a limit of the separation between points, the directional derivative reads

$$\begin{aligned} \partial_u f &:= u \lrcorner df = df(u) \\ &= u^i(x)\, \frac{\partial f(x)}{\partial x^i} \qquad \text{in local coordinates } \{x^i\}. \end{aligned} \tag{C.1.1}$$ directU

Obviously, (C.1.1) describes a map $\partial_u : T^0_0(X) \times T^1_0(X) \to T^0_0(X)$ of scalar fields, i.e., tensors of type $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, again into $T^0_0(X) = C(X)$. This map has simple properties: 1) $\mathbb{R}$-linearity, 2) $C(X)$-linearity with respect to u, i.e., $\partial_{gu+hv} f = g\partial_u f + h\partial_v f$, 3) additivity $\partial_u(f+g) = \partial_u f + \partial_u g$.

In order to generalize the directional derivative to arbitrary tensor fields of type $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$, one needs a recipe of how to compare tensor quantities at two different points of a manifold. This is provided by an additional structure on X, called the *linear connection*. The linear connection or, equivalently, the covariant differentiation is necessary in order to formulate differential equations for various physical fields like, for instance, the Einstein equation for the gravitational field or the Navier-Stokes equation of hydrodynamics.

In line with the directional derivative, a *covariant differentiation* ∇ is defined as an smooth $\mathbb{R}$-linear map

$$\nabla : T^p_q(X) \times T^1_0(X) \longrightarrow T^p_q(X) \tag{C.1.2}$$ CDdef

which to any vector field $u \in T^1_0(X)$ and to any tensor field $T \in T^p_q(X)$ of type $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ assigns a tensor field $\nabla_u T \in T^p_q(X)$ of type $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ that satisfies the following properties:

1) $C(X)$-linearity with respect to u,

$$\nabla_{fu+gv}T = f\nabla_u T + g\nabla_v T\,, \tag{C.1.3}$$ codiff1

2) additivity with respect to T,

$$\nabla_u(T+S) = \nabla_u T + \nabla_u S\,, \tag{C.1.4}$$ codiff2

3) for a scalar field f a directional derivative is recovered,

$$\nabla_u f = u \lrcorner \mathrm{d}f\,, \tag{C.1.5}$$ codiff3

4) the Leibniz rule with respect to tensor product,

$$\nabla_u(T\otimes S) = \nabla_u T\otimes S + T\otimes\nabla_u S\,, \tag{C.1.6}$$ codiff4

5) the Leibniz rule with respect to interior product,

$$\nabla_u(v\lrcorner\omega) = (\nabla_u v)\lrcorner\omega + v\lrcorner\nabla_u\omega\,, \tag{C.1.7}$$ codiff5

for all vector fields u, v, all functions f, g, all tensor fields T, S, and all forms ω.

The Lie derivative $\mathcal{L}_u$, defined in Sec. A.2.10, is also a map $T^p_q(X)\times T^1_0(X)\to T^p_q(X)$. The properties of covariant differentiation 3)-6) are the same as those of the Lie derivative, cf. (C.1.4)-(C.1.7) with (A.2.56), (A.2.53), (A.2.58) and (A.2.55), respectively. The property (C.1.3) is however somewhat different than the corresponding property of the Lie derivative (A.2.57) which can be visualized, e.g., by substituting $u\to fu$ in (A.2.57). This difference reflects the fact that the definition of $\mathcal{L}_u T$ at a given point $x\in X$ makes use of u in the neighborhood of this point, whereas in order to define $\nabla_u T$ at x one has to know only the value of u at x.

C.1.2 Linear connection 1-forms

The difference between the covariant and the partial differentiation of a tensor field is determined by the linear connection 1-forms $\Gamma_i{}^j$. They show up in the action of ∇_u on a frame ∂_i and supply a constructive realization of the covariant differentiation ∇_u of an arbitrary tensor field.

Consider the chart U_1 with the local coordinates x^i which contains a point $x \in U_1 \subset X$. Take the natural basis ∂_i at a x. The covariant differentiation ∇_u of the vectors ∂_i with respect to an arbitrary vector field $u = u^k \partial_k$ reads

$$\nabla_u \partial_i = u^k \nabla_{\partial_k} \partial_i. \tag{C.1.8}$$

Here we used the property (C.1.3). These n vector fields can be decomposed with respect to the coordinate frame ∂_i:

$$\nabla_u \partial_i := \Gamma_i{}^j(u)\, \partial_j. \tag{C.1.9}$$

conn1

The connection 1-forms

$$\Gamma_i{}^j = \Gamma_{ki}{}^j\, dx^k \tag{C.1.10}$$

conn2

can be read off with their components $\Gamma_{ki}{}^j$. Suppose a different chart U_2 with the local coordinates $x^{i'}$ intersects with U_1, and the point $x \in U_1 \bigcap U_2$ belongs to an intersection of the two charts. Then the connection 1-forms satisfy the consistency condition

$$\Gamma_{i'}{}^{j'}(x') = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \Gamma_i{}^j(x) + \frac{\partial x^{j'}}{\partial x^k}\, d\, \frac{\partial x^k}{\partial x^{i'}} \tag{C.1.11}$$

coordcon

everywhere in the intersection of the local charts $U_1 \bigcap U_2$.

Now, making use of the properties 1)-5), one can describe a covariant differentiation of an arbitrary $\left[{p \atop q}\right]$ tensor field

$$T = T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q} \tag{C.1.12}$$

arbtensor

in terms of the connection 1-forms. Namely, we have explicitly the $\begin{bmatrix} p \\ q \end{bmatrix}$ tensor field

$$\nabla_u T = u \rfloor (DT^{i_1\dots i_p}{}_{j_1\dots j_q})\partial_{i_1} \otimes \dots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}\,, \tag{C.1.13}$$ coderiv

where the *covariant differential* of the natural tensor components is introduced by

$$\begin{aligned} D\,T^{i_1\dots i_p}{}_{j_1\dots j_q} := d\,(T^{i_1\dots i_p}{}_{j_1\dots j_q}) \\ + \Gamma_k{}^{i_1}\,T^{k i_2\dots i_p}{}_{j_1\dots j_q} + \dots + \Gamma_k{}^{i_p}\,T^{i_1\dots i_{p-1}k}{}_{j_1\dots j_q} \\ - \Gamma_{j_1}{}^k\,T^{i_1\dots i_p}{}_{k j_2\dots j_q} - \dots - \Gamma_{j_q}{}^k\,T^{i_1\dots i_p}{}_{j_1\dots j_{q-1}k}\,. \end{aligned} \tag{C.1.14}$$ coderiv2

The step in (C.1.9) can be generalized to an arbitrary frame e_α. Its covariant differentiation with respect to a vector field u reads

$$\nabla_u e_\alpha = \Gamma_\alpha{}^\beta(u) e_\beta\,, \tag{C.1.15}$$ connect

with the corresponding linear connection 1-forms $\Gamma_\alpha{}^\beta$. In 4 dimensions, we have 16 one-forms $\Gamma_\alpha{}^\beta$ at our disposal. The components of the connection 1-forms with respect to the coframe ϑ^α are given by

$$\Gamma_\alpha{}^\beta = \Gamma_{\gamma\alpha}{}^\beta \vartheta^\gamma \qquad \text{or} \qquad \nabla_{e_\alpha} e_\beta = \Gamma_{\alpha\beta}{}^\gamma e_\gamma\,. \tag{C.1.16}$$ comps

In terms of a local coordinate system $\{x^i\}$,

$$\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta \mathrm{d}x^i \qquad \text{where} \qquad \Gamma_{i\alpha}{}^\beta = \Gamma_\alpha{}^\beta(\partial_i)\,. \tag{C.1.17}$$ holcomps

The 1-forms $\Gamma_\alpha{}^\beta$ are not a new independent object: since an arbitrary frame may be decomposed with respect to the coordinate frame, we find, with the help of (A.2.30) and (A.2.31), the simple relation

$$\Gamma_\alpha{}^\beta = e_j{}^\beta\,\Gamma_i{}^j\,e^i{}_\alpha + e_i{}^\beta\,d\,e^i{}_\alpha. \tag{C.1.18}$$ gamgam

Under a change of the frame which is described by a linear transformation

$$e_{\alpha'} = L_{\alpha'}{}^\alpha e_\alpha\,, \tag{C.1.19}$$ chframe

the connection 1-forms transform in a non-tensorial way,

$$\Gamma_{\alpha'}{}^{\beta'} = L_{\alpha'}{}^{\alpha} L_{\beta}{}^{\beta'} \Gamma_{\alpha}{}^{\beta} + L_{\gamma}{}^{\beta'} \mathrm{d} L_{\alpha'}{}^{\gamma} \,. \qquad \text{(C.1.20)}$$ trafocon

For an infinitesimal linear transformation, $L_\alpha{}^\beta = \delta^\beta_\alpha + \varepsilon_\alpha{}^\beta$,

$$\delta e_\alpha = e_{\alpha'} - e_\alpha = \varepsilon_\alpha{}^\beta e_\beta, \qquad \text{(C.1.21)}$$ chframe1

the connection one-form changes as

$$\delta\Gamma_\alpha{}^\beta = D\varepsilon_\alpha{}^\beta = d\varepsilon_\alpha{}^\beta + \Gamma_\gamma{}^\beta\,\varepsilon_\alpha{}^\gamma - \Gamma_\alpha{}^\gamma \varepsilon_\gamma{}^\beta. \qquad \text{(C.1.22)}$$

Although the transformation law (C.1.20) is inhomogeneous, we cannot, on an open set U, achieve $\Gamma_{\alpha'}{}^{\beta'} = 0$ in general; it can be only done if the curvature (to be introduced later) vanishes in U. At a given point x_0, however, we can always choose the first derivatives of $L_{\alpha'}{}^\alpha$ contained in $\mathrm{d}L_{\alpha'}{}^\alpha$ in such a way that $\Gamma_{\alpha'}{}^{\beta'}(x_0) = 0$. A frame e_α, such that

$$\Gamma_\alpha{}^\beta(x_0) = 0 \qquad \text{at a given point } x_0\,, \qquad \text{(C.1.23)}$$ normframe

will be called *normal* at x_0. A normal frame is given up to transformations $L_{\alpha'}{}^\alpha$ whose first derivatives vanish at x_0. This freedom may be used, and we can always choose a coordinate system $\{x^i\}$ such that the frame $e_\alpha(x)$ in (C.1.23) is also 'trivialized':

$$e_\alpha = \delta^i_\alpha \partial_i \qquad \text{at a given point } x_0\,. \qquad \text{(C.1.24)}$$ normcoo

Summing up[2], for a *trivialized frame* we have

$$(e_\alpha, \Gamma_\alpha{}^\beta) \stackrel{*}{=} (\delta^i_\alpha \partial_i, 0) \qquad \text{at a given point } x_0\,. \qquad \text{(C.1.25)}$$ trivial

Despite the fact that the normal frame looks like a coordinate frame [in the sense, e.g., that (C.1.24) shows apparently that the tangent vectors ∂_i of the coordinate frame coincide with the vectors of e_α basis at x_0], one cannot, in general, introduce new *local coordinates* in the neighborhood of x_0 in which (C.1.23) is fulfilled.

[2]See Hartley [3] and Iliev [4].

C.1.3 ⊗Covariant differentiation of a general geometric quantity

What is the covariant differential of a tensor density, for example?

Let us now consider the covariant differentiation of a general geometric quantity that was introduced in Sec. A.1.3. As earlier in Sec. A.2.10, we will treat a geometric quantity w as a set of smooth fields w^A on X. These fields are the components of $w = w^A e_A$ with respect to a frame $e_A \in W = \mathbb{R}^N$ in the space of a ρ-representation of the group $GL(n, \mathbb{R})$ of local linear frame transformations (C.1.19). The transformation (C.1.19) of a frame of spacetime acts on the geometric quantity of type ρ by the means of the local generalization of (A.1.18),

$$w^A \longrightarrow w^{A'} \;=\; \rho_B{}^{A'}(L^{-1})\, w^B, \qquad \text{(C.1.26)}$$

or, in the infinitesimal case (C.1.21),

$$\delta\, w^A = -\,\varepsilon_\alpha{}^\beta\, \rho_B{}^{A\,\alpha}{}_\beta\, w^B. \qquad \text{(C.1.27)}$$ geomtrafo

The generator matrix $\rho_B{}^{A\,\alpha}{}_\beta$ was introduced in (A.2.67) when we discussed the Lie derivative of geometric quantities of type ρ.

A covariant differentiation for geometric quantities of type ρ is introduced as a natural generalization of the map (C.1.2) with all the properties 1)-6) preserved: the covariant differentiation for a W-frame reads

$$\nabla_u e_A = \rho_A{}^{B\,\alpha}{}_\beta\, \Gamma_\alpha{}^\beta(u) e_B\,, \qquad \text{(C.1.28)}$$ connect2

whereas for an arbitrary geometric quantity $w = w^A\, e_A$ of type ρ eqs. (C.1.13), (C.1.14) are replaced by

$$\nabla w = u \,\rfloor\, (Dw^A)\, e_A, \quad \text{with} \quad Dw^A := dw^A + \rho_B{}^{A\,\alpha}{}_\beta\, \Gamma_\alpha{}^\beta\, w^B. \qquad \text{(C.1.29)}$$ coderiv3

The general formula (C.1.29) is consistent with the covariant derivative of usual tensor fields when the latter are treated as a geometric quantity of a special kind; one can compare this with the examples 2), 3) in Sec. A.1.3.

Two simple applications of this general technique are in order. As a first one, we recall that the Levi-Civita symbols have the same values with respect to all frames, see (A.1.65). This means that they are the geometric quantities of the type $\rho = \mathrm{id}$ (identity transformation) or, plainly speaking, $\delta\epsilon^{\alpha_1\cdots\alpha_n} = 0$. Comparing this with (C.1.27) and (C.1.28), we conclude that

$$D\,\epsilon^{\alpha_1\ldots\alpha_n} = 0 \qquad \text{(C.1.30)}$$ Deps0

for an arbitrary connection.

As a second example, let us take a scalar density $\mathcal{S}$ of weight w. This geometric quantity is described by the transformation law (A.1.57) or, in the infinitesimal form, by

$$\delta\mathcal{S} = w\,\varepsilon_\alpha{}^\alpha\,\mathcal{S}. \qquad \text{(C.1.31)}$$

Comparing with (C.1.27), we find $\rho^\alpha{}_\beta = -\,w\,\delta^\alpha_\beta$, and hence (C.1.29) yields

$$D\,\mathcal{S} = d\mathcal{S} - w\,\Gamma_\alpha{}^\alpha\,\mathcal{S}. \qquad \text{(C.1.32)}$$

If we generalize this to a tensor density , $\mathcal{T}_{\alpha\ldots}{}^{\beta\cdots}$, we find

$$D\,\mathcal{T}_{\alpha\ldots}{}^{\beta\cdots} = d\,\mathcal{T}_{\alpha\ldots}{}^{\beta\cdots} - \Gamma_\alpha{}^\mu\,\mathcal{T}_{\mu\ldots}{}^{\beta\cdots} - \cdots + \Gamma_\mu{}^\beta\,\mathcal{T}_{\alpha\ldots}{}^{\mu\cdots} + \cdots - w\,\Gamma_\mu{}^\mu\,\mathcal{T}_{\alpha\ldots}{}^{\beta\cdots}. \qquad \text{(C.1.33)}$$ density

C.1.4 Parallel transport

By means of the covariant differentiation ∇_u, a tensor can be parallelly transported along a curve on a manifold. This provides a convenient tool for comparing values of the tensor field at different points of the manifold.

A connection enables us to define *parallel transport* of a tensor along a curve. A differentiable curve σ on X is a smooth map

$$\begin{aligned} \sigma\ :\ &(a,b)\to X \\ &t\ \ \mapsto x(t)\,, \end{aligned} \qquad \text{(C.1.34)}$$

where (a, b) is an interval in $\mathbb{R}$. In local coordinates $\{x^i\}$, the tangent vector to the curve $\sigma = \{x^i(t)\}$ is $u^i = dx^i/dt$. A tensor field T is said to be *parallelly transported* along σ if

$$\nabla_u T = 0 \qquad \text{(C.1.35)}$$ nabla0

along σ. Taking into account (C.1.14), we get for T

$$\frac{d}{dt} T^{i_1 \dots i_p}{}_{j_1 \dots j_q}\big(x(t)\big) + \frac{dx^l}{dt} \Bigg(\Gamma_{lk}{}^{i_1}(x(t))\, T^{k i_2 \dots i_p}{}_{j_1 \dots j_q}(x(t)) \; + \; \dots \\ - \; \Gamma_{l j_q}{}^{k}(x(t))\, T^{i_1 \dots i_p}{}_{j_1 \dots j_{q-1} k}(x(t)) \Bigg) = 0\,. \qquad \text{(C.1.36)}$$ paralT

If $0 \in \ (a, b)$ and $T(\sigma(0))$ is given, then there exists (at least locally) a unique solution $T(\sigma(t))$ of this linear ordinary differential equation for $t \in \ (a, b)$. Therefore, we have a linear map of tensors of type $\left[{p \atop q}\right]$ at the point $\sigma(0)$ to tensors of type $\left[{p \atop q}\right]$ at the point $\sigma(t)$.

Taking $T = u$, we obtain a differential equation for *autoparallels*:

$$\nabla_u u = 0 \qquad \text{or} \qquad \frac{d^2 x^i}{dt^2} + \Gamma_{jk}{}^{i}(x(t))\, \frac{dx^j}{dt} \frac{dx^k}{dt} = 0. \qquad \text{(C.1.37)}$$ autopar

C.1.5 $^{\otimes}$Torsion and curvature

Torsion and curvature both measure the deviation from the flat spacetime geometry (of special relativity). When both of them are zero, one can globally define the trivialized frame (C.1.25) all over the spacetime manifold.

We now want to associate with a connection two tensor fields: torsion and curvature. As we have seen above, the connection form can always be transformed to zero at one given point. However, torsion and curvature will in general give a non-vanishing characterization of the connection at this point.

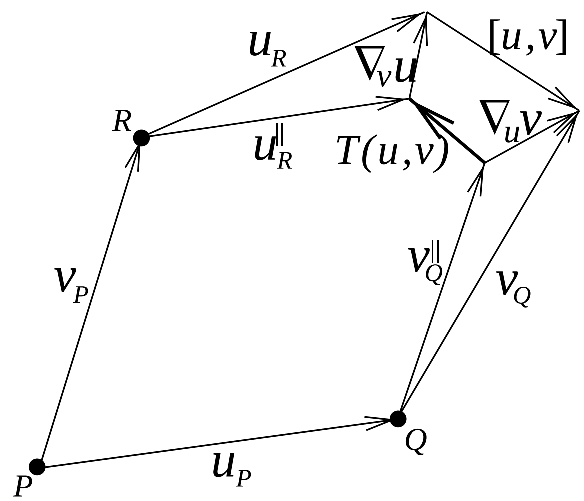

Figure C.1.1: On the geometrical interpretation of torsion: a closure failure of infinitesimal displacements.This is a schematic view. Note that R and Q are infinitesimally near to P.

The *torsion* of a connection ∇ is a map T that assigns to each pair of vector fields u and v a vector field $T(u,v)$ by

$$T(u,v) := \nabla_u v - \nabla_v u - [u,v]\,. \tag{C.1.38}$$ deftorsion

The commutator $[u,v]$ has been defined in (A.2.5). One can straightforwardly verify the tensor character of $T(u,v)$ so that its value at any given point is determined by the values of u and v at that point.

Fig. C.1.1 illustrates schematically the geometrical meaning of torsion: Choose two vectors v and u at a point $P \in X$. Transfer u parallelly along v to the point R and likewise v along u to the point Q. If the resulting parallelogram is *broken*, i.e., if it has a gap or a closure failure, then the connection carries a torsion. Such situations occur in the continuum theory of dislocations.[3]

[3] See Kröner [5] and references given there.

Since the torsion $T(u,v)$ is a vector field, one can expand it with respect to a local frame,

$$T(u,v) = T^{\alpha}(u,v)\, e_{\alpha}\,. \tag{C.1.39}$$ torexp

By construction, the coefficients $T^{\alpha}(u,v)$ of this expansion are are 2-forms, i.e., functions which assign real numbers to every pair of the vector fields u, v. Taking the vectors of a frame,

$$T(e_{\beta}, e_{\gamma}) = T^{\alpha}(e_{\beta}, e_{\gamma})\, e_{\alpha} = T_{\beta\gamma}{}^{\alpha}\, e_{\alpha}\,, \tag{C.1.40}$$ torcoef

we can read off the coefficients of this *vector-valued torsion 2-form*:

$$T^{\alpha} = \frac{1}{2}\, T_{\mu\nu}{}^{\alpha}\, \vartheta^{\mu}\wedge\vartheta^{\nu} = \frac{1}{2}\, T_{ij}{}^{\alpha}\, dx^{i}\wedge dx^{j}\,. \tag{C.1.41}$$ tor2form

The explicit form of these coefficients is obtained directly from the definition (C.1.38) which we evaluate with respect to a frame e_{α},

$$T^{\alpha}(e_{\beta}, e_{\gamma})\, e_{\alpha} = \nabla_{e_{\beta}}\, e_{\gamma} - \nabla_{e_{\gamma}}\, e_{\beta} - [e_{\beta},\, e_{\gamma}]^{\alpha}\, e_{\alpha}. \tag{C.1.42}$$

The first two terms on the right-hand side bring in the connection coefficients (C.1.16), whereas the commutator can be rewritten in terms of the object of anholonomity, see (A.2.36). Accordingly, we find for the components of the torsion

$$T_{\beta\gamma}{}^{\alpha} = \Gamma_{\beta\gamma}{}^{\alpha} - \Gamma_{\gamma\beta}{}^{\alpha} + C_{\beta\gamma}{}^{\alpha}. \tag{C.1.43}$$ torcomps

The torsion 2-form T^{α} allows to define the 1-form $e_{\alpha}\rfloor T^{\beta}$. If added to the connection 1-form $\Gamma_{\alpha}{}^{\beta}$, it is again a connection. We call it the *transposed connection*

$$\begin{aligned} \widehat{\Gamma}_{\alpha}{}^{\beta} := \Gamma_{\alpha}{}^{\beta} + e_{\alpha}\rfloor T^{\beta} &= \left(\Gamma_{\gamma\alpha}{}^{\beta} + T_{\gamma\alpha}{}^{\beta}\right)\vartheta^{\gamma} \\ &= \left(\Gamma_{\alpha\gamma}{}^{\beta} + C_{\alpha\gamma}{}^{\beta}\right)\vartheta^{\gamma}, \end{aligned} \tag{C.1.44}$$ transconn

since in a natural frame, i.e. for $C_{\alpha\gamma}{}^{\beta} = 0$, the indices of the components of $\widehat{\Gamma}_{\alpha}{}^{\beta}$ are transposed with respect to that of $\Gamma_{\alpha}{}^{\beta}$.

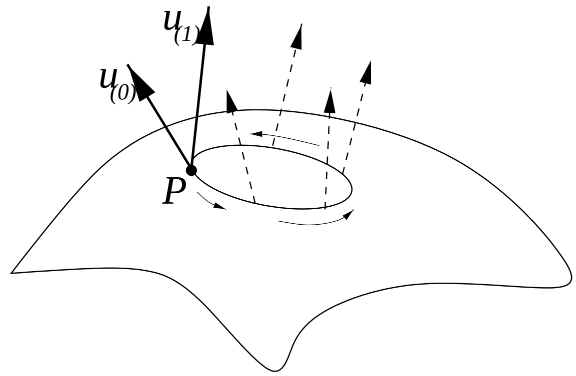

Figure C.1.2: On the geometrical interpretation of curvature: parallel transport of a vector around a closed loop.

The *curvature* of a connection ∇ is a map that assigns to each pair of vector fields u and v a linear transformation $R(u,v): X_x \to X_x$ of the tangent space at an arbitrary point x by

$$R(u,v)w := \nabla_u \nabla_v\, w - \nabla_v \nabla_u\, w - \nabla_{[u,v]}\, w\,. \qquad \text{(C.1.45)}$$ defcurv

One can verify the tensor character of the curvature so that the value of $R(u,v)w$ at any given point depends only on the values of vector fields u, v, and w at that point.

Analogously to torsion, we can expand the four vector fields $R(u,v)e_\alpha$ with respect to a local vector frame,

$$R(u,v)\, e_\alpha = R_\alpha{}^\beta(u,v)\, e_\beta\,. \qquad \text{(C.1.46)}$$ curvexp

Thereby the curvature 2-form $R_\alpha{}^\beta$ is defined. Similarly to (C.1.41), we may express these 2-forms with respect to a coframe ϑ^α or a coordinate frame dx^i as follows:

$$R_\alpha{}^\beta = \frac{1}{2} R_{\mu\nu\alpha}{}^\beta\, \vartheta^\mu \wedge \vartheta^\nu = \frac{1}{2} R_{ij\alpha}{}^\beta\, dx^i \wedge dx^j\,. \qquad \text{(C.1.47)}$$ curv2form

Taking (C.1.46) with respect to a frame, one finds the components of the curvature 2-form,

$$R(e_\mu, e_\nu)e_\alpha = R_{\mu\nu\alpha}{}^\beta\, e_\beta \quad \text{and} \quad R(\partial_i, \partial_j)e_\alpha = R_{ij\alpha}{}^\beta\, e_\beta\,. \qquad \text{(C.1.48)}$$

Thus, by (C.1.45),

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} - \partial_\nu \Gamma_{\mu\alpha}{}^{\beta} + \Gamma_{\mu\sigma}{}^{\beta}\Gamma_{\nu\alpha}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\beta}\Gamma_{\mu\alpha}{}^{\sigma} + C_{\mu\nu}{}^{\sigma}\Gamma_{\sigma\alpha}{}^{\beta} \tag{C.1.49}$$

curv2

and

$$R_{ij\alpha}{}^{\beta} = \partial_i \Gamma_{j\alpha}{}^{\beta} - \partial_j \Gamma_{i\alpha}{}^{\beta} + \Gamma_{i\sigma}{}^{\beta}\Gamma_{j\alpha}{}^{\sigma} - \Gamma_{j\sigma}{}^{\beta}\Gamma_{i\alpha}{}^{\sigma}\,. \tag{C.1.50}$$

curvcomp

Here we introduced the abbreviation $\partial_\mu := e^i{}_\mu\, \partial_i$.

The curvature 2-form $R_\alpha{}^\beta$ can be contracted by means of the frame e_β. In this way we find the Ricci 1-form

$$\mathrm{Ric}_\alpha := e_\beta \rfloor R_\alpha{}^\beta = \mathrm{Ric}_{\alpha\beta}\, \vartheta^\beta. \tag{C.1.51}$$

ricci1

Using (C.1.47), we immediately find

$$\mathrm{Ric}_{\alpha\beta} = R_{\gamma\alpha\beta}{}^{\gamma}. \tag{C.1.52}$$

ricci2

The geometrical meaning of the curvature is revealed when we consider a parallel transport of a vector along a closed curve in X, see Fig. C.1.2. Let $\sigma : \{x^i(t)\},\ 0 \le t \le 1$, be a smooth curve which starts and ends at a point $P = x^i(0) = x^i(1)$ [in other words, σ is a 1-cycle]. Taking a vector $u_{(0)}$ at P and transporting it parallelly along σ [which technically reduces to the solution of a differential equation (C.1.36)], one finds at the return point $x^i(1)$ a vector $u_{(1)}$ which differs from $u_{(0)}$. The difference is determined by the curvature,

$$\Delta\, u^\alpha \;=\; u^\alpha_{(1)} - u^\alpha_{(0)} = -\int\limits_S R_\beta{}^\alpha\, u^\beta, \tag{C.1.53}$$

intcurv

where S is the two-dimensional surface which is bounded by σ, i.e., $\sigma = \partial S$.

When the curvature is zero everywhere in X, we call such a manifold a *flat affine space with torsion* (or a teleparallelism spacetime). If the torsion vanishes additionally, we speak of a flat affine space.

Affine means that the connection ∇ is still there and it allows to compare tensors at different points. Clearly, the curvature is vanishing for an everywhere trivial connection form $\Gamma_\alpha{}^\beta = 0$. However, as we know, the components $\Gamma_\alpha{}^\beta$ depend on the frame field and, in general, if we have curvature, it may happen to be impossible to choose frames e_α in such a way that $\Gamma_\alpha{}^\beta = 0$ on the whole X. On a flat affine space, the components of a vector do not change after a parallel transport around a closed loop, in other words, parallel transport is integrable. Curvature is a measure of the deviation from the flat case.

C.1.6 ⊗Cartan's geometric interpretation of torsion and curvature

On a manifold with a linear connection, the notion of a position vector can be defined along a curve. If we transport the position vector around an infinitesimal closed loop, it is subject to a translation and a linear transformation. The translation reveals the torsion and the linear transformation the curvature of the manifold.

Let us start, following Cartan, with the flat affine space, in which a connection ∇ has zero torsion and zero curvature. In such a manifold, we may define an affine *position vector field* (or radius vector field) r as one that satisfies the equation

$$\nabla_v r = v \tag{C.1.54}$$ affinetrans

for all vector fields v. With respect to a local coordinates $\{x^i\}$, $r = r^i\,\partial_i$, and (C.1.54) is a system of sixteen partial differential equations for the four functions $r^i(x)$, namely

$$D\,r^j = dx^i, \qquad \text{or} \qquad \partial_i\,r^j(x) + \Gamma_{ik}{}^j(x)\,r^k(x) = \delta_i^j . \tag{C.1.55}$$ affinetrans1

In flat affine space, a coordinate basis can always be chosen, at least within one chart, in such a way that $\Gamma_{ik}{}^j = 0$, and then the equation (C.1.54) or (C.1.55) is simply $\partial_j r^i = \delta^i_j$. The solution is $r^i = x^i + A^i$, where A^i a constant vector, so that r^i is, indeed, the position (or radius) vector of x^i with respect to an origin $x^i = -A^i$.

In an affine spacetime, the integrability condition for (C.1.54) is

$$\nabla_v\nabla_w r - \nabla_w\nabla_v\,r - \nabla_{[v,w]}r = \nabla_v w - \nabla_w v - [v,w]\,, \tag{C.1.56}$$

or

$$R(v,w)r - T(v,w) = 0\,, \tag{C.1.57}$$

for all vector fields v, w. Hence a sufficient condition for the existence of global radius vector fields r is

$$R(v,w) = 0 \qquad \text{and} \qquad T(v,w) = 0 \tag{C.1.58}$$

for arbitrary v, w, i.e. the vanishing curvature and torsion.

In a general manifold, when torsion and curvature are non–zero, the position vector field does not exist on X. Nevertheless, it is possible to define a position vector *along a curve*. This object turns out to be extremely useful for revealing the geometrical meaning of torsion and curvature.

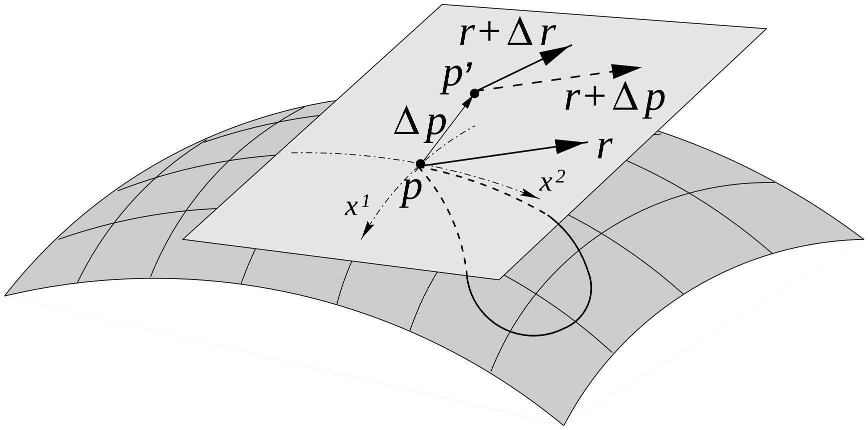

Figure C.1.3: Cartan's position vector on manifold with curvature and torsion: Affine transport of a vector r around an infinitesimal loop. Here p and p' denote the initial and final positions of the position vector in the affine tangent space.

Let $p \in X$ be an arbitrary point, and $\sigma = \{x^i(t)\},\ t \geq 0$, be a smooth curve which starts at p, i.e. $x^i(0) = x^i_p$. We now attach at p an affine tangent space, or, to put it differently, we will consider the tangent space X_p at p as an n-dimensional *affine vector space*. Recall that in an affine vector space, each element (vector) is given by its origin and its components to some fixed basis. We will construct Cartan's position vector as a map which assigns to each point of a curve σ a vector in the affine tangent space X_p. Geometrically such a construction can be conveniently understood as a generalized development map which is defined by 'rolling' the tangent space along a given curve.

The defining equation of the position vector is again (C.1.54), but this time v is not an arbitrary vector field but tangent to the curve under consideration, i.e. $v = (dx^i/dt)\partial_i$. Substituting this into (C.1.54), we get

$$\frac{dr^\alpha(t)}{dt} = \left[e_i^\alpha(x(t)) - \Gamma_{i\beta}{}^\alpha(x(t))\, r^\beta(t)\right] \frac{dx^i}{dt}\,. \qquad \text{(C.1.59)}$$

With the growing of t, one 'moves' along the curve σ and the functions $r^\alpha(t)$ always describe the components of the position vector in the affine tangent space at the *fixed original* point p. Thus, for example, a displacement along the curve from x^i to $x^i + d\xi^i$ yields the change of the position vector

$$dr^\alpha = e_i{}^\alpha(x)\, d\xi^i - \Gamma_{i\beta}{}^\alpha(x) r^\beta(x)\, d\xi^i. \qquad \text{(C.1.60)}$$ changeu

Following Cartan, we may interpret this equation as telling us that the position vector map consists of a *translation* $e_i^\alpha\, d\xi^i$ and a *linear transformation* $-\Gamma_{i\beta}{}^\alpha\, d\xi^i\, r^\beta$ in the affine tangent space at p.

Let us now consider a closed curve σ, i.e. such that $x^i(1) = x^i_p$, see Fig. C.1.3. Then, on integrating around σ, it is found that the total change in r^α is given by

$$\Delta r^\alpha = \int_S (T^\alpha - R_\beta{}^\alpha\, r^\beta) = (T_{ij}{}^\alpha - R_{ij\beta}{}^\alpha\, r^\beta) \int_S dx^i \wedge dx^j\,, \qquad \text{(C.1.61)}$$ totalchange

where S is the two–dimensional infinitesimal surface enclosed by σ. Thus, in going around the infinitesimal closed loop σ, the position vector r in the affine tangent space at p undergoes a translation and a linear transformation, of the same order of magnitude as the area of S. The translation is determined by the torsion

$$\Delta p^\alpha = \int_S T^\alpha , \tag{C.1.62}$$

whereas the linear transformation is determined by the curvature [it is instructive to compare this with the change of a vector under the parallel transport (C.1.53)].

C.1.7 ⊗Covariant exterior derivative

If an exterior form is generalized to a tensor-valued *exterior form, then the usual definition of the exterior derivative can be naturally extended to the* covariant *exterior derivative. Covariant exterior derivatives of torsion and curvature are involved in the two Bianchi identities.*

Torsion 2-form (C.1.39), and (C.1.41) and curvature 2-form (C.1.46), and (C.1.47) are examples of tensor-valued p-forms, that is, of generalized geometric quantities. For such objects we need to introduce the notion of covariant exterior derivative which shares the properties of a covariant derivative of a geometric quantity and of an exterior derivative of a scalar-valued form.

Let φ^A be an arbitrary p-form of type ρ. It can be written as sum of decomposable p-forms of type ρ, namely $\varphi^A = w^A\,\omega$ where w^A is a scalar of type ρ and ω a usual exterior p-form. For such a form we define

$$D\varphi^A := (\nabla w^A)\,\omega + w^A\,d\omega \tag{C.1.63}$$

and extend this definition by $\mathbb{R}$-linearity to arbitrary p-forms of type ρ. Using (C.1.29), it is straightforward to obtain the general formula

$$\boxed{D\varphi^A = d\varphi^A + \rho_B{}^{A\,\alpha}{}_\beta\,\Gamma_\alpha{}^\beta \wedge \varphi^B} \tag{C.1.64}$$ Drhomega

and to prove that D satisfies the Leibniz rule

$$D(\varphi^A \wedge \psi^B) = D\varphi^A \wedge \psi^B + (-1)^p \varphi^A \wedge D\psi^B\,, \qquad \text{(C.1.65)}$$

where p is the degree of φ^A.

Unlike the usual exterior derivative, which satisfies $dd = 0$, the covariant exterior derivative is no longer nilpotent:

$$DD\varphi^A = \rho_B{}^{A\alpha}{}_\beta\, R_\alpha{}^\beta \wedge \varphi^B\,. \qquad \text{(C.1.66)}$$ riccident

The simplest proof makes use of the normal frame (C.1.23) in which $D\varphi^A \stackrel{*}{=} d\varphi^A$, $R_\alpha{}^\beta \stackrel{*}{=} d\Gamma_\alpha{}^\beta$. We choose a normal frame and differentiate (C.1.64). Since the resulting formula is an equality of two $(p+2)$-forms of type ρ, it holds in an arbitrary frame. The relation (C.1.66) is called the *Ricci identity.*

Now we can appreciable simplify all calculations involving frame, connection, curvature, and torsion. At first, noticing that the coframe ϑ^α is a 1-form of the vector type, we recover the torsion 2-form (C.1.39), (C.1.41) as a covariant exterior derivative

$$T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta\,. \qquad \text{(C.1.67)}$$ structure1

This equation is often called the first (Cartan) structure equation. Applying (C.1.66), we obtain the *1st Bianchi identity*:

$$\boxed{DT_\alpha = R_\beta{}^\alpha \wedge \vartheta^\beta}\,. \qquad \text{(C.1.68)}$$ bianchi1

Analogously, after recognizing the curvature 2-form as a generalized 2-form of tensor type $\left[{1 \atop 1}\right]$, we immediately rewrite (C.1.46), (C.1.47) as

$$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta + \Gamma_\sigma{}^\beta \wedge \Gamma_\alpha{}^\sigma, \qquad \text{(C.1.69)}$$ structure2

which is called the second (Cartan) structure equation. Using again the trick with the normal frame, we obtain the *2nd Bianchi identity*:

$$\boxed{DR_\alpha{}^\beta = 0}\,. \qquad \text{(C.1.70)}$$ bianchi2

Finally, we can link up the notions of Lie derivative and covariant derivative. For decomposable p-forms of type ρ, $\varphi^A = w^A\,\omega$, where w^A is a scalar of type ρ and ω a p-form, we define the *covariant* Lie derivative as

$$Ł_u\varphi^A := (\nabla_u w^A)\,\omega + w^A\mathcal{L}_u\omega \qquad \text{(C.1.71)}$$

and extend this definition by $\mathbb{R}$-linearity to arbitrary p-forms of type ρ. It is an interesting exercise to show that

$$Ł_u\varphi^A = u \rfloor D\varphi^A + D(u \rfloor \varphi^A)\,. \qquad \text{(C.1.72)}$$ covariantLie

C.1.8 The p-forms o(a), conn1(a,b), torsion2(a), curv2(a,b)

We come back to our Excalc programming. We put $n = 4$. On each differential manifold, we can specify an arbitrary coframe field ϑ^α, in Excalc `o(a)`. Excalc is made familiar with `o(a)` by means of the coframe statement as described in Sec. A.2.11. Moreover, since we introduced a linear connection 1-form $\Gamma_\alpha{}^\beta$, we do the same in Excalc with `pform conn1(a,b)=1;` then it is straightforward to implement the torsion and curvature 2-forms T^α and $R_\alpha{}^\beta$ by means of the structure equations (C.1.67) and (C.1.69), respectively:

```
pform torsion2(a)=2, curv2(a,b)=2;
                          % preceded by coframe command
torsion2(a):=d o(a) + conn1(-b,a)^o(b);
curv2(-a,b):=d conn1(-a,b) + conn1(-a,c)^conn1(-c,b);
```

In Excalc, the trace of the torsion $T := e_\alpha \rfloor T^\alpha$ reads `e(-a) _| torsion2(a)`, and the corresponding trace part of the torsion $T^\alpha = \frac{1}{3}\,\vartheta^\alpha \wedge T$ becomes

```
pform trator2(a);
trator2(a) :=o(a)^(e(-a) _|torsion2(a));
```

The Ricci 1-form is encoded as

```
pform ricci1(a)=1;
ricci1(-a) :=e(-b) _|curv2(-a,b);
```

Weyl's (purely non-Riemannian) dilational (or segmental) curvature 2-form $\frac{1}{4}\,\delta_\alpha{}^\beta\,R_\gamma{}^\gamma$ is the other generally covariant contraction of the curvature. We have

```
pform delta(a,b)=0, dilcurv2(a,b)=2;

delta(-0,0:=delta(-1,1):=delta-3,3):=delta(-1,1):=1;
dilcurv2(-a,b):=delta(-a,b)*curv2(-c,c)/4;
```

These are the quantities which play a role in a 4-dimensional differential manifold with a prescribed connection. The corresponding Excalc expressions defined here can be put into an executable Excal program. However, first we want to get access to a possible metric of this manifold.

C.2
Metric

Although in our axiomatic discussion of electrodynamics in Part B we adhered to the connection-free and metric-free point of view, the notions of connection and metric are unavoidable in the end. In the previous Chapter C.1, we gave the fundamentals of the geometry of a manifolds equipped with a linear connection. Here we discuss the metric.

In Special Relativity theory (SR) and in the corresponding classical field theory in flat spacetime, the Lorentzian metric enters as a fundamental *absolute* element. In particular, all physical particles are defined in terms of representations of the Poincaré (or inhomogeneous Lorentz) group which has a metric built in from the very beginning.

In General Relativity theory (GR), the metric field is upgraded to the status of a *gravitational potential*. In particular, the Einstein field equation is formulated in terms of a Riemannian metric with Lorentz signature carrying on its right-hand side the symmetric (Hilbert or metric) energy-momentum tensor as a material source. The physical significance of the spacetime metric lies in the fact that it determines intervals between events in spacetime ds^2, and, furthermore, establishes the causal structure of spacetime.

It is important to realize that the two geometrical structures — connection and metric — a priori are absolutely independent from each other. Modern data convincingly demonstrate the validity of Riemannian geometry and Einstein's GR on macroscopic scales where mass (energy-momentum) of matter alone determines the structure of spacetime. However, at high energies, the properties of matter are significantly different, with additional spacetime related characteristics, such as spin and scale charge coming into play. Correspondingly, one can expect that the geometric structure of spacetime on small distances may deviate from Riemannian geometry.

"In the dilemma whether one should ascribe to the world primarily a metric or an affine structure, the best point of view may be the neutral one which treats the g's as well as the Γ's as independent state quantities. Then the two sets of equations, which link them together, become laws of nature without attributing a preferential status as definitions to one or the other half." [1]

C.2.1 Metric vector spaces

A metric tensor introduces the length of a vector and an angle between every two vectors. The components of the metric are defined by the values of the scalar products of the basis vectors.

Let us consider a linear vector space V. It is called a *metric vector space* if on V a scalar product is defined as a bilinear

[1] "In dem Dilemma, ob man der Welt ursprünglich eine metrische oder eine affine Struktur zuschreiben soll, ist vielleicht der beste Standpunkt der neutrale, der sowohl die g wie die Γ als unabhängige Zustandsgrößen behandelt. Dann werden die beiden Sätze von Gleichungen, welche sie verbinden, zu Naturgesetzen ohne daß die eine oder andere Hälfte als Definitionen eine bevorzugte Stellung bekommen." H. Weyl: *50 Jahre Relativitätstheorie* [9], our translation.

symmetric and non-degenerate map

$$\mathbf{g} : V \times V \longrightarrow \mathbb{R}. \tag{C.2.1}$$

In other words, a scalar product is introduced by a *metric tensor* $\mathbf{g}$ of type $\left[{0 \atop 2}\right]$ which is symmetric, i.e. $\mathbf{g}(u,v) = \mathbf{g}(v,u)$ for all $u,\, v \in V$, and non-degenerate in the sense that $\mathbf{g}(u,v) = 0$ holds for all v if and only if $u = 0$. The real number

$$\mathbf{g}(u,u) \tag{C.2.2}$$

is called a *length* of a vector u. The metric $\mathbf{g}$ defines a canonical isomorphism of the vector space and its dual,

$$\tilde{\mathbf{g}} : V \longrightarrow V^*, \tag{C.2.3}$$
isoVV

where the 1-form $\tilde{\mathbf{g}}(u)$, if applied to a vector v, yields

$$\tilde{\mathbf{g}}(u)\,(v) := \mathbf{g}(u,v), \quad \text{for all } v \in V\,. \tag{C.2.4}$$
gtilde

Alternatively, we may write $\tilde{\mathbf{g}}(u) = \mathbf{g}(u,\,\cdot\,)$.

In terms of a basis e_α of V and the dual basis ϑ^α of V^*,

$$\mathbf{g} = g_{\alpha\beta}\,\vartheta^\alpha \otimes \vartheta^\beta, \quad \text{with} \quad g_{\alpha\beta} := \mathbf{g}(e_\alpha, e_\beta) = g_{\beta\alpha}\,. \tag{C.2.5}$$
metcoeff

Thus the isomorphism (C.2.3) is technically reduced to the vertical motion of indices,

$$\tilde{\mathbf{g}}(e_\alpha)\,(e_\beta) = \mathbf{g}(e_\alpha, e_\beta) = g_{\alpha\beta} = g_{\alpha\gamma}\,\delta^\gamma_\beta = g_{\alpha\gamma}\,\vartheta^\gamma(e_\beta)\,. \tag{C.2.6}$$

Accordingly, the basis vectors define the 1-forms via

$$\tilde{\mathbf{g}}(e_\alpha) = g_{\alpha\gamma}\,\vartheta^\gamma =: \vartheta_\alpha\,. \tag{C.2.7}$$
gtildinv

Under a change of the basis (A.1.5), the metric coefficients $g_{\alpha\beta}$ transform according to (A.1.11). Recall that a symmetric matrix can always be brought into a diagonal form by a linear transformation. A basis for which

$$g_{\alpha\beta} = \mathrm{diag}\,(1, \ldots, 1, -1, \ldots, -1) \tag{C.2.8}$$
orthomet

is called *orthonormal.* We will mainly be interested in 4-dimensional spacetime. Its tangent vector space at each event is Minkowskian. Therefore, from now on let us take V to be a 4-dimensional Minkowskian vector space, unless specified otherwise. The components of the metric tensor with respect to an orthonormal basis are then given by

$$g_{\alpha\beta} = o_{\alpha\beta} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad \text{(C.2.9)}$$ oij

C.2.2 $^{\otimes}$Orthonormal, half-null, and null frames, the coframe statement

A null vector has zero length. A set of null vectors is in many cases a convenient tool for the construction of a special basis in a Minkowski vector space.

The Minkowski metric has many interesting 'faces' which we will mention here only briefly.

Traditionally, in relativity theory the vectors of an *orthonormal* basis are labeled by $0, 1, 2, 3$, thus underlining the fundamental difference between e_0, which has a positive length $g_{00} = \mathbf{g}(e_0, e_0) = 1$, and e_a, $a = 1, 2, 3$, which have negative length $g_{aa} = \mathbf{g}(e_a, e_a) = -1$. In general, a vectors $u \in V$ is called *time-like* if $\mathbf{g}(u, u) < 0$, *space-like* if $\mathbf{g}(u, u) > 0$, and *null* if $\mathbf{g}(u, u) = 0$.

In Excalc one specifies the coframe as the primary quantity. If we use Cartesian coordinates, an orthonormal coframe and frame in Minkowski space read, respectively,

```
coframe o(0) = d t ,
        o(1) = d x ,
        o(2) = d y ,
        o(3) = d z  with
  metric  g  = o(0)*o(0)-o(1)*o(1)-o(2)*o(2)-o(3)*o(3);
frame e;
```

The blank between `d` and `t` etc. is necessary! Note that the phrase `with metric g=o(0)*o(0)-o(1)*o(1)-o(2)*o(2)-o(3)*o(3);` in this case of a diagonal metric, can also be abbreviated by `with signature 1,-1,-1,-1;`

We recall that, in Excalc, a specific spherically symmetric coframe in a 4-dimensional Riemannian spacetime with Lorentzian signature has already been defined in Sec. B.5.6 in our Maxwell sample program. In general relativity, for the gravitational field of a mass m and angular momentum per unit mass a, one has axially symmetric metrics with coframes like

```
pform   rr=0, delsqrt=0, ffsqrt=0$
fdomain rr=rr(rho,theta),delsqrt=delsqrt(rho),
        ffsqrt=ffsqrt(theta)$

coframe
o(0)=(delsqrt/rr)*(d t-(a0*sin(theta)**2)*d phi),
o(1)=(ffsqrt/rr)*sin(theta)*(a0*d t-(rho**2+a0**2)*d phi),
o(2)=(rr/ffsqrt)*d theta,
o(3)=(rr/delsqrt)*d rho
with signature 1,-1,-1,-1$
```

Here `rr, delsqrt, ffsqrt` are functions to be determined by the Einstein equation. This is an example of coframe that is a bit more involved.

Starting from an orthonormal basis e_α with respect to which the metric has the standard form (C.2.9), we can build a new frame $e_{\alpha'} = (l, n, e_2, e_3)$ by the linear transformation:

$$l = \frac{1}{\sqrt{2}}(e_0 + e_1), \qquad n = \frac{1}{\sqrt{2}}(e_0 - e_1), \tag{C.2.10}$$

and $e_{2'} = e_2, e_{3'} = e_3$. The first two vectors of the new frame are null: $\mathbf{g}(l,l) = \mathbf{g}(n,n) = 0$. Correspondingly, the metric in this *half-null* basis reads

$$g_{\alpha\beta} \;= h_{\alpha\beta} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{C.2.11}$$ `halfnull`

In Excalc, again with Cartesian coordinates, we find

```
coframe h(0) = (d t+d x)/sqrt(2),
        h(1) = (d t-d x)/sqrt(2),
        h(2) =  d y,
        h(3) =  d z     with
  metric  hh = h(0)*h(1)+h(1)*h(0)-h(2)*h(2)-h(3)*h(3);
```

You can convince yourself by `displayframe;` and `on nero; hh(-a,-b);` that all has been understood by Excalc correctly.

Following Newman & Penrose, we can further construct two more *null* vectors as the *complex* linear combinations of e_2 and e_3:

$$m = \frac{1}{\sqrt{2}}(e_2 + i\,e_3), \qquad \overline{m} = \frac{1}{\sqrt{2}}(e_2 - i\,e_3). \tag{C.2.12}$$

Here i is the imaginary unit, and overbar means the complex conjugation. This transformation leads to the Minkowski metric in a *null* (Newman-Penrose) basis $e_{\alpha'} = (l, n, m, \overline{m})$:

$$g_{\alpha\beta} \;= n_{\alpha\beta} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{C.2.13}$$ NPnull

Such a basis is convenient for investigating the properties of gravitational and electromagnetic waves. In Excalc, we have

```
coframe n(0) = (d t +   d x)/sqrt(2),
        n(1) = (d t -   d x)/sqrt(2),
        n(2) = (d y + i*d z)/sqrt(2),
        n(3) = (d y - i*d z)/sqrt(2)  with
  metric  nn = n(0)*n(1)+n(1)*n(0)-n(2)*n(3)-n(3)*n(2);
```

In the Newman & Penrose frame, we have two real null legs, namely l and n, and two complex ones, m and $\overline{m}$. It may be surprising to learn that it is also possible to define the *null symmetric* frame of D. Finkelstein which consists of four real null vectors. We start from an orthonormal basis e_α, with $\mathbf{g}(e_\alpha, e_\beta) = o_{\alpha\beta}$, and define the new basis f_α according to

$$
\begin{aligned}
f_0 &= (\sqrt{3}\, e_0 + e_1 + e_2 + e_3)/2, \\
f_1 &= (\sqrt{3}\, e_0 + e_1 - e_2 - e_3)/2, \\
f_2 &= (\sqrt{3}\, e_0 - e_1 + e_2 - e_3)/2, \\
f_3 &= (\sqrt{3}\, e_0 - e_1 - e_2 + e_3)/2.
\end{aligned}
\qquad \text{(C.2.14)}
$$

f2e

Since $\mathbf{g}(f_\alpha, f_\alpha) = 0$ for all α, the null symmetric frame consists solely of real non-orthogonal null-vectors. The metric with respect to this frame reads

$$
g_{\alpha\beta} = f_{\alpha\beta} := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \qquad \text{(C.2.15)}
$$

finkmat

The metric (C.2.15) looks completely symmetric in all its components: Seemingly the time coordinate is not preferred in any sense. Nevertheless (C.2.15) is a truly Lorentzian metric. Its determinant is -3 and the eigenvalues are readily computed to be

$$
3, \quad -1, \quad -1, \quad -1, \qquad \text{(C.2.16)}
$$

which shows that the metric (C.2.15) has, indeed, the correct signature.

There is a beautiful geometrical interpretation of the four null legs of the Finkelstein frame. In a Minkowski spacetime, let us consider the three-dimensional spacelike hypersurface which is spanned by (e_1, e_2, e_3). The four points which are defined by the spatial parts of the Finkelstein basis vectors (C.2.14), with coordinates $A = (1, 1, 1)$, $B = (1, -1, -1)$, $C = (-1, 1, -1)$, and $D = (-1, -1, 1)$, form a perfect tetrahedron in the 3-subspace. The vertices A, B, C, and D lie at the same distances of equal to $\sqrt{3}$ from the origin $O = (0, 0, 0)$ (and correspondingly all sides of this tetrahedron have equal length, namely $\sqrt{8}$). If we

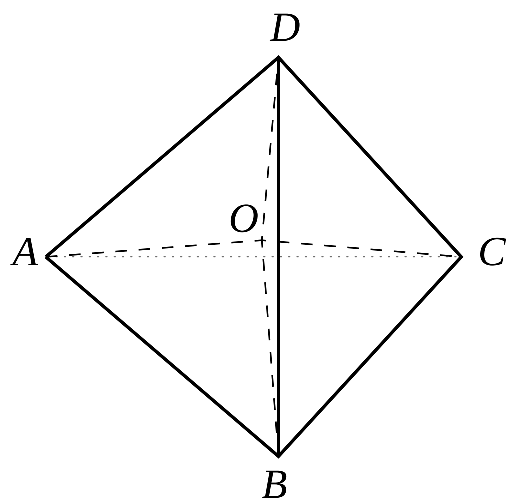

Figure C.2.1: Tetrahedron which defines Finkelstein basis

now send, at the moment $t = 0$, a light pulse from the origin O, it reaches all four vertices of the tetrahedron at $t = \sqrt{3}$. Thus four light rays provide the operational definition for the *light-like* Finkelstein basis (C.2.14).

C.2.3 Metric volume 4-form

> *Given a metric, a corresponding orthonormal coframe determines a metric volume 4-form on every vector space.*

Let ϑ^α be an orthonormal coframe in the four-dimensional Minkowski vector space $(V^*, \mathbf{g})$. Let us define the product

$$\eta = \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 \,. \tag{C.2.17}$$

eta4

If $\vartheta^{\alpha'}$ is another orthonormal coframe in $(V^*, \mathbf{g})$, then

$$\vartheta^{\alpha'} = L_\alpha{}^{\alpha'}\, \vartheta^\alpha \,, \tag{C.2.18}$$

where the transformation matrix is (pseudo)orthogonal, i.e.,

$$o_{\alpha'\beta'}\, L_\alpha{}^{\alpha'} L_\beta{}^{\beta'} = o_{\alpha\beta} \quad \text{and} \quad L := \det\left(L_\alpha{}^{\alpha'}\right) = \pm 1 \,. \tag{C.2.19}$$

Therefore, under this change of the basis, we have

$$\vartheta^{0'} \wedge \vartheta^{1'} \wedge \vartheta^{2'} \wedge \vartheta^{3'} = L\, \vartheta^{0} \wedge \vartheta^{1} \wedge \vartheta^{2} \wedge \vartheta^{3} . \tag{C.2.20}$$

In other words, the definition (C.2.17) gives us a unique (i.e. basis independent) *twisted volume 4-form* of the Minkowski vector space. Alternatively, one may consider two non-twisted volume 4-forms separately for each orientation.

If $\vartheta^{\alpha'}$ is an *arbitrary* (not necessarily orthonormal) frame, then we have $\vartheta^{\alpha'} = L_{\alpha}{}^{\alpha'} \vartheta^{\alpha}$ and – since η is twisted –

$$\eta = |L|\, \vartheta^{0'} \wedge \vartheta^{1'} \wedge \vartheta^{2'} \wedge \vartheta^{3'} . \tag{C.2.21}$$

On the other hand $e_{\alpha'} = L_{\alpha'}{}^{\alpha}\, e_{\alpha}$. Thus, from the tensor transformation $g_{\alpha'\beta'} = L_{\alpha'}{}^{\alpha} L_{\beta'}{}^{\beta}\, o_{\alpha\beta}$, we obtain $|L|^2 = -\det(g_{\alpha'\beta'})$. Hence the twisted volume element with respect to the frame $e_{\alpha'}$ reads

$$\eta = \sqrt{-\det(g_{\alpha'\beta'})}\, \vartheta^{0'} \wedge \vartheta^{1'} \wedge \vartheta^{2'} \wedge \vartheta^{3'} . \tag{C.2.22}$$ eta

Dropping the primes in (C.2.22), we may write, for any basis ϑ^{α}, the metric volume 4-form as

$$\boxed{\eta := \frac{1}{4!}\, \eta_{\alpha\beta\gamma\delta}\, \vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \vartheta^{\gamma} \wedge \vartheta^{\delta}}\,, \tag{C.2.23}$$ eta1

where the twisted antisymmetric tensor $\eta_{\alpha\beta\gamma\delta}$ of type $[{0 \atop 4}]$ is defined by

$$\eta_{\alpha\beta\gamma\delta} := \sqrt{-\det(g_{\mu\nu})}\, \hat{\epsilon}_{\alpha\beta\gamma\delta} , \tag{C.2.24}$$

and $\hat{\epsilon}_{\alpha\beta\gamma\delta}$ is the Levi-Civita permutation symbol with $\hat{\epsilon}_{0123} = +1$. If we raise the indices in the usual way, we find the contravariant components as

$$\eta^{\alpha\beta\gamma\delta} = g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma}\, \eta_{\mu\nu\rho\sigma} = -\frac{1}{\sqrt{-\det(g_{\mu\nu})}}\, \epsilon^{\alpha\beta\gamma\delta} . \tag{C.2.25}$$ eta2

It follows from (C.2.25) that

$$\vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \vartheta^{\gamma} \wedge \vartheta^{\delta} = -\eta^{\alpha\beta\gamma\delta}\, \eta . \tag{C.2.26}$$

C.2.4 Duality operator for 2-forms as a symmetric almost complex structure on M^6

The duality operator grows out of the almost complex structure **J** *on the* M^6 *when* **J** *is additionally self-adjoint with respect to the natural metric on the* M^6*.*

Let us now turn again to the space M^6 of the 2-forms in four dimensions which we discussed in Sec. A.1.10.

When an almost complex structure **J** is introduced in M^6 in such a way that it is additionally *symmetric*, i.e. self-adjoint, with respect to the natural metric $\boldsymbol{\varepsilon}$ (A.1.89) on M^6, then a linear operator on 2-forms

$$^{\#} : \Lambda^2 V^* \longrightarrow \Lambda^2 V^* \,, \tag{C.2.27}$$

dualdef1

defined by means of $^{\#} = \mathbf{J}$, is called the *duality operator* on M^6. In view of (A.1.109), the duality operator satisfies

$$^{\#\#} = -1. \tag{C.2.28}$$

This will be called the *closure relation* of the duality operator. The *self-adjointness* with respect to the 6-metric (A.1.89) means

$$\boldsymbol{\varepsilon}(^{\#}\omega,\, \varphi) = \boldsymbol{\varepsilon}(\omega,\, ^{\#}\varphi) \tag{C.2.29}$$

symsharp

for all $\omega, \varphi \in M^6$.

The explicit action of the duality operator on the basis 2-forms reads

$$^{\#}\left(\vartheta^\alpha \wedge \vartheta^\beta\right) = \frac{1}{2}\, \#_{\rho\sigma}{}^{\alpha\beta} \left(\vartheta^\rho \wedge \vartheta^\sigma\right), \tag{C.2.30}$$

sharp2B

where the elements of the matrix $\#_{\rho\sigma}{}^{\alpha\beta}$ are, by definition, the components of the almost complex structure in M^6. In the 6-dimensional notation introduced in Sec. A.1.10, the duality operator is thus described by

$$^{\#}\mathcal{B}^K = \#_I{}^K\, \mathcal{B}^I. \tag{C.2.31}$$

sharp4

Expressed in terms of components, the metric reads $\boldsymbol{\varepsilon}(\omega,\varphi) = \varepsilon^{IJ}\omega_I\varphi_J$, see (A.1.90). Therefore, the self-adjointness can also be written as $\varepsilon^{IJ}\#_I{}^K\omega_K\varphi_J = \varepsilon^{IJ}\#_J{}^K\omega_I\varphi_K$, or

$$\overset{\circ}{\chi}{}^{IJ} = \overset{\circ}{\chi}{}^{JI} \qquad \text{for} \qquad \overset{\circ}{\chi}{}^{IJ} := \varepsilon^{IK}\#_K{}^J. \tag{C.2.32}$$

chisym

By construction, the components of the duality matrix can be expressed in terms of the 3×3 blocks imported from (A.1.110),

$$\#_I{}^K = \begin{pmatrix} C^a{}_b & A^{ab} \\ B_{ab} & C^b{}_a \end{pmatrix}. \tag{C.2.33}$$

sharpIJ

Here the components are constrained by the self-adjointness (C.2.29),

$$A^{ab} = A^{ba}, \qquad B_{ab} = B_{ba}, \qquad C^a{}_a = 0, \tag{C.2.34}$$

sharp5

as compared to the almost complex structure (in particular, the D-block is expressed in terms of C). Besides that, the algebraic condition (A.1.111) is replaced by the closure relation

$$A^{ac}B_{cb} + C^a{}_cC^c{}_b = -\delta^a_b, \quad C^{(a}{}_cA^{b)c} = 0, \quad C^c{}_{(a}B_{b)c} = 0. \tag{C.2.35}$$

sharpclose

The existence of the duality operator has an immediate consequence for the complexified space $M^6(\mathbb{C})$ of 2-forms. As we saw in Sec. A.1.11, the almost complex structure provides for a splitting of the $M^6(\mathbb{C})$ into the 2 three-dimensional subspaces corresponding to the $\pm i$ eigenvalues of $\mathbf{J}$. Now we can say even more: These two subspaces are orthogonal to each other in the sense of the natural 6-metric (A.1.89):

$$\boldsymbol{\varepsilon}(\omega,\varphi) = 0, \qquad \text{for all} \qquad \omega \in \overset{\text{(s)}}{M},\ \varphi \in \overset{\text{(a)}}{M}. \tag{C.2.36}$$

ort

The proof is straightforward: $\boldsymbol{\varepsilon}(\omega,\varphi) = i\boldsymbol{\varepsilon}(\omega,{}^{\#}\varphi) = i\boldsymbol{\varepsilon}({}^{\#}\omega,\varphi) = -\boldsymbol{\varepsilon}(\omega,\varphi)$, where we used the definitions (A.1.113) and the symmetry property (C.2.29).

The 6-metric $\boldsymbol{\varepsilon}$ induces a metric on the three-dimensional subspaces (A.1.113), turning them into the complex Euclidean 3-spaces. The symmetry group, which preserves the induced 3-metric on $\overset{(s)}{M}$ (and on $\overset{(a)}{M}$), is $SO(3,\mathbb{C})$. This is a group-theoretic origin of the reconstruction of the spacetime metric from the duality operator: the Lorentz group, being isomorphic to $SO(3,\mathbb{C})$, is encoded in the structure of the self-dual (or, equivalently, anti-self-dual) complex 2-forms on X.

The significance of the duality operator will become clearer in the next sections where we will explicitly demonstrate that $^{\#}$ enables us to construct a Lorentzian metric on spacetime.

C.2.5 From the duality operator to a triplet of complex 2-forms

Every duality operator on M^6 determines a triplet of complex 2-forms that satisfy certain completeness conditions.

Suppose the duality operator (C.2.27) is defined in the M^6 with the closure property (A.1.109) and the self-adjointness (C.2.29). Its action on the basis 2-forms is given by (C.2.31), with the matrix (C.2.33), (C.2.34). Using the natural $3+3$ split of the two-form basis (A.1.81), eq.(C.2.31) is rewritten, with the help of (C.2.33), as

$$^{\#}\begin{pmatrix}\beta\\ \hat{\epsilon}\end{pmatrix} = \begin{pmatrix} C & A\\ B & C^T\end{pmatrix}\begin{pmatrix}\beta\\ \hat{\epsilon}\end{pmatrix}, \qquad \text{(C.2.37)} \quad \texttt{dualB}$$

or, in terms of its matrix elements,

$$\begin{aligned} ^{\#}\beta^a &= C^a{}_b\,\beta^b + A^{ab}\,\hat{\epsilon}_b,\\ ^{\#}\hat{\epsilon}_a &= B_{ab}\,\beta^b + C^b{}_a\,\hat{\epsilon}_b. \end{aligned} \qquad \text{(C.2.38)} \quad \texttt{dualuv}$$

By means of the duality operator $^{\#}$ (as well as with $\mathbf{J}$ of the almost complex structure), one can decompose any 2-form into a *self-dual* and an *anti-self-dual* part. In terms of the 2-form basis, this reads,

$$B^I = \overset{(s)}{B}{}^I + \overset{(a)}{B}{}^I, \tag{C.2.39}$$

where we define

$$\overset{(s)}{B}{}^I := \frac{1}{2}(B^I \; - \; i\,{}^{\#}B^I), \tag{C.2.40}$$ scb

$$\overset{(a)}{B}{}^I := \frac{1}{2}(B^I \; + \; i\,{}^{\#}B^I). \tag{C.2.41}$$ acb

Here i is the imaginary unit. One can check that

$$\begin{aligned} {}^{\#}\overset{(s)}{B}{}^I &= +i\,\overset{(s)}{B}{}^I, \\ {}^{\#}\overset{(a)}{B}{}^I &= -i\,\overset{(a)}{B}{}^I. \end{aligned} \tag{C.2.42}$$ sacb

Thus the 6-dimensional space of complex 2-forms $M^6(\mathbb{C})$ decomposes into two 3-dimensional invariant subspaces which correspond to the two eigenvalues $\pm i$ of the duality operator.

In order to construct the bases of these subspaces, we have to inspect the 3+3 representation (A.1.81). One has, using (A.1.81) in (C.2.40), the two sets of the self-dual 2-forms,

$$\overset{(s)}{\beta}{}^a = \frac{1}{2}(\beta^a \; - \; i\,{}^{\#}\beta^a), \tag{C.2.43}$$

$$\overset{(s)}{\epsilon}{}_a = \frac{1}{2}(\hat{\epsilon}_a \; - \; i\,{}^{\#}\hat{\epsilon}_a), \tag{C.2.44}$$

and similarly for the anti-self-dual forms. With the help of (C.2.38), we find explicitly:

$$\overset{(s)}{\beta}{}^a = \frac{1}{2}\left[(\delta^a_b - iC^a{}_b)\,\beta^b - iA^{ab}\,\hat{\epsilon}_b\right], \tag{C.2.45}$$ Sbet

$$\overset{(s)}{\epsilon}{}_a = \frac{1}{2}\left[(\delta^b_a - iC^b{}_a)\,\hat{\epsilon}_b - iB_{ab}\,\beta^b\right]. \tag{C.2.46}$$ Sgam

Since the invariant subspace of the self-dual forms is 3-dimensional, these two triplets of self-dual forms cannot be independent from each other. Indeed, let us multiply (C.2.45) with B_{ca} and (C.2.46) with $C^a{}_c$. Then the sum of the resulting relation, with the help of the closure property (C.2.35), yields

$$B_{ac}\overset{(s)}{\beta}{}^a = (i\delta^a_c - C^a{}_c)\overset{(s)}{\epsilon}{}_a \,. \tag{C.2.47}$$

This shows that these two triplets are linearly dependent. For the non-degenerate B-matrix, one can express $\overset{(s)}{\beta}{}^a$ in terms of $\overset{(s)}{\epsilon}{}_a$ explicitly.

Let us compute the exterior products of the triplets (C.2.45) and (C.2.46):

$$\overset{(s)}{\beta}{}^a \wedge \overset{(s)}{\beta}{}^b = -\frac{1}{2}\left(iA^{ab} + C^{(a}{}_c A^{b)c}\right)\text{Vol} = -\frac{i}{2}\,A^{ab}\,\text{Vol}, \tag{C.2.48}$$

$$\overset{(s)}{\epsilon}{}_a \wedge \overset{(s)}{\epsilon}{}_b = -\frac{1}{2}\left(iB_{ab} + C^c{}_{(a}B_{b)c}\right)\text{Vol} = -\frac{i}{2}\,B_{ab}\,\text{Vol}. \tag{C.2.49}$$

We used here the algebra (A.1.85)-(A.1.87) and the closure property (C.2.35). Furthermore, we have

$$\overset{(s)}{\beta}{}^a \wedge \overline{\overset{(s)}{\beta}}{}^b = \frac{1}{2}\,C^{(a}{}_c A^{b)c}\,\text{Vol} = 0, \tag{C.2.50}$$

$$\overset{(s)}{\epsilon}{}_a \wedge \overline{\overset{(s)}{\epsilon}}{}_b = \frac{1}{2}C^c{}_{(a}B_{b)c}\,\text{Vol} = 0. \tag{C.2.51}$$

Here the overbar denotes complex conjugate objects.

C.2.6 From the triplet of complex 2-forms to a duality operator

Every triplet of complex 2-forms with the completeness property determine a duality operator on M^6.

Both (C.2.45) and (C.2.46) are particular representations of the following general structure: Given is a *triplet of self-dual complex 2-forms* $S^{(a)}$ such that

$$S^{(a)} \wedge S^{(b)} = -2i\,G^{ab}\,\text{Vol}, \tag{C.2.52}$$ SSG

$$S^{(a)} \wedge \overline{S}^{(b)} = 0, \tag{C.2.53}$$ SOS

where the matrix G^{ab} is *real and non-degenerate*. Overbar denotes complex conjugate objects. Equivalently, one can rewrite (C.2.52) as

$$S^{(a)} \wedge S^{(b)} = \frac{1}{3} G^{ab} \, G_{cd} \, S^{(c)} \wedge S^{(d)}, \tag{C.2.54}$$

where G_{ab} denotes the matrix inverse to G^{ab}. We will call (C.2.52), (C.2.53) the *completeness* conditions for a triplet of 2-forms.

In the previous section we saw that every duality operator defines a triplet of self-dual complex 2-forms. Here we show that the converse is also true: Let $S^{(a)}$ be an arbitrary triplet of complex 2-forms which satisfies the completeness conditions (C.2.52), (C.2.53). Then they determine a duality operator in M^6.

Expanding the arbitrary 2-forms with respect to the basis (A.1.81), we can write

$$S^{(a)} = M^a{}_b \, \beta^b + N^{ab} \, \hat{\epsilon}_b. \tag{C.2.55}$$ matS

In view of (A.1.85)-(A.1.87), we find that (C.2.52) imposes an algebraic constraint on the matrix components,

$$M^a{}_c \, N^{bc} + M^b{}_c \, N^{ac} = -2G^{ab}. \tag{C.2.56}$$ SSG0

Introducing the real variables

$$M^a{}_b = V^a{}_b + i \, U^a{}_b, \qquad N^{ab} = X^{ab} + iY^{ab}, \tag{C.2.57}$$

one can decompose (C.2.56) into the two real equations:

$$V^{(a}{}_c \, X^{b)c} - U^{(a}{}_c \, Y^{b)c} = 0, \tag{C.2.58}$$ SSG1

$$V^{(a}{}_c \, Y^{b)c} + U^{(a}{}_c \, X^{b)c} = -G^{ab}. \tag{C.2.59}$$ SSG2

Analogously, (C.2.53) yields another pair of the real matrix equations:

$$V^{(a}{}_c \, X^{b)c} + U^{(a}{}_c \, Y^{b)c} = 0, \tag{C.2.60}$$ SOS1

$$V^{[a}{}_c \, Y^{b]c} - U^{[a}{}_c \, X^{b]c} = 0. \tag{C.2.61}$$ SOS2

Combining (C.2.58) and (C.2.60), we find

$$V^{(a}{}_c X^{b)c} = 0, \quad U^{(a}{}_c Y^{b)c} = 0, \tag{C.2.62}$$ VWXY1

whereas the sum of (C.2.59) and (C.2.61) gives

$$V^a{}_c Y^{bc} + U^b{}_c X^{ac} = -G^{ab}. \tag{C.2.63}$$ VWXY2

We can count the number of independent degrees of freedom. The total number of variables is 36 (= 4×9 unknown components of the matrices V, U, X, Y). They are subject to 21 constraints (=6 + 6 + 9) imposed by the equations (C.2.62) and (C.2.63). Thus, in general, 15 degrees are left over for the unknown matrices in (C.2.55).

In order to see how the triplet is related to the duality operator, let us define a new basis in the M^6 by means of the linear transformation

$$\begin{pmatrix} \beta'^a \\ \hat{\epsilon}'_a \end{pmatrix} = \begin{pmatrix} V^a{}_b & X^{ab} \\ -U_{ab} & -Y_a{}^b \end{pmatrix} \begin{pmatrix} \beta^b \\ \hat{\epsilon}_b \end{pmatrix}, \tag{C.2.64}$$ B2B-S

where $U_{ab} := G_{ac}U^c{}_b$ and $Y_a{}^b := G_{ac}Y^{cb}$. The new basis elements satisfy the same algebraic conditions

$$\beta'^a \wedge \beta'^b = 0, \quad \hat{\epsilon}'_a \wedge \hat{\epsilon}'_b = 0, \quad \hat{\epsilon}'_a \wedge \beta'^b = \delta^b_a \, \mathrm{Vol} \tag{C.2.65}$$

as those in (A.1.85)-(A.1.87). The proof follows directly from (C.2.62) and (C.2.63). Interestingly, the transformation (C.2.64) is *always* invertible, with the inverse given by

$$\begin{pmatrix} \beta^a \\ \hat{\epsilon}_a \end{pmatrix} = \begin{pmatrix} -Y_b{}^a & X^{ba} \\ -U_{ba} & V^b{}_a \end{pmatrix} \begin{pmatrix} \beta'^b \\ \hat{\epsilon}'_b \end{pmatrix}. \tag{C.2.66}$$ B2B-I

The direct check again involves only the completeness conditions (C.2.62) and (C.2.63).

The original triplet (C.2.55), with respect to the new basis (C.2.64), then reduces to

$$S^{(a)} = \beta'^a - iG^{ab}\hat{\epsilon}'_b\,. \tag{C.2.67}$$ matSnew

Now we are prepared to introduce the duality operator. We define it by simply postulating that its action on the triplet amounts to a mere multiplication by the imaginary unit, i.e.,

$$^{\#}S^{(a)} = i\,S^{(a)} \qquad \left(\text{hence} \qquad ^{\#}\overline{S}^{(a)} = -\,i\,\overline{S}^{(a)}\right). \qquad \text{(C.2.68)}$$

From (C.2.67) we have $\beta'^{a} = (S^{(a)}+\overline{S}^{(a)})/2$ and $\hat{\epsilon}'_{a} = iG_{ab}(S^{(a)} - \overline{S}^{(a)})/2$. Then, immediately, we find the action of the duality operator on the new 2-form basis:

$$\# \begin{pmatrix} \beta'^{a} \\ \hat{\epsilon}'_{a} \end{pmatrix} = \begin{pmatrix} 0 & G^{ab} \\ -G_{ab} & 0 \end{pmatrix} \begin{pmatrix} \beta'^{b} \\ \hat{\epsilon}'_{b} \end{pmatrix}. \qquad \text{(C.2.69)}$$ newdual

We can now reconstruct the original basis in (C.2.37). We use (C.2.66) and (C.2.64) and find the 3×3 matrices

$$A^{ab} = G_{cd}\left(Y^{ac}Y^{bd} - X^{ac}X^{bd}\right), \qquad \text{(C.2.70)}$$ A3S

$$B_{ab} = G_{cd}\left(U^{c}{}_{a}U^{d}{}_{b} - V^{c}{}_{a}V^{d}{}_{b}\right), \qquad \text{(C.2.71)}$$ B3S

$$C^{a}{}_{b} = G_{cd}\left(U^{c}{}_{b}Y^{da} - V^{c}{}_{b}X^{da}\right). \qquad \text{(C.2.72)}$$ C3S

As a consequence, every triplet of complex 2-forms, which satisfy the completeness conditions (C.2.52) and (C.2.53), defines a duality operator with the closure and the symmetry properties.

C.2.7 From a triplet of complex 2-forms to the metric: Schönberg-Urbantke formulas

The triplet of complex 2-forms is a building material for the metric of spacetime. The Lorentzian metric, up to a scale factor, can be constructed from the triplet of 2-forms by means of the Schönberg-Urbantke formulas.

The importance of the duality operator $^{\#}$ and the corresponding triplet of 2-forms lies in the fact that they determine a Lorentzian metric on 4-space.

Let us formulate this result. Let a triplet of two-forms $S^{(a)}$ be given on V which satisfy (C.2.52) with some symmetric regular matrix G. Then the Lorentzian metric of spacetime is recovered with the help of the Schönberg-Urbantke formulas:

$$\sqrt{\det g}\, g_{ij} = \frac{16}{3}\sqrt{\det G}\,\epsilon^{klmn}\,\hat{\epsilon}_{abc}\, S^{(a)}_{ik} S^{(b)}_{lm} S^{(c)}_{nj}\,, \tag{C.2.73}$$

urbantke1

$$\sqrt{\det g} = \frac{1}{24}\,\epsilon^{klmn}\, G_{cd}\, S^{(c)}_{kl} S^{(d)}_{mn}. \tag{C.2.74}$$

urbantke2

Here, ϵ^{klmn} is the Levi-Civita symbol, and $S^{(a)}_{ij}$ are the components of the basis two-forms with respect to the local coordinates $\{x^i\}$, i.e., $S^{(a)} = \frac{1}{2} S^{(a)}_{ij} dx^i \wedge dx^j$. Despite the appearance of the Levi-Civita symbols in (C.2.73)-(C.2.74), these expressions are tensorial.

A rigorous proof of the fact that on a 4-dimensional vector space V every three complex two-forms $S^{(a)}$, which satisfy the completeness condition (C.2.52), define a (pseudo-)Riemannian metric will not be presented here[2]. The metric is, in general, complex, however it is real when (C.2.53) is fulfilled.

Note that the sign in the algebraic equations (A.1.109) and (C.2.35) is important. If, instead of the minus, there appeared a plus, then the resulting spacetime metric would have a Riemannian (Euclidean) signature $(+1, +1, +1, +1)$ or a mixed one $(+1, +1, -1, -1)$.

As a comment to the Schönberg-Urbantke mechanism, we would like to mention the local isomorphism of the three following complex Lie groups (symplectic, special linear, and orthogonal):

$$Sp(1, \mathbb{C}) \approx SL(2, \mathbb{C}) \approx SO(3, \mathbb{C}). \tag{C.2.75}$$

The easiest way to see this is to analyze the corresponding Lie algebras. At first, recall that the orthogonal algebra $so(3, \mathbb{C})$ consists of all skew-symmetric 3×3 matrices with complex elements:

$$a = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}. \tag{C.2.76}$$

so3mat

[2]See, however, Schönberg [7], Urbantke [8], or Harnett [2].

Here q_1, q_2, q_3 is a triplet of complex numbers.

Since the linear algebra $sl(2,\mathbb{C})$ consists of all traceless 2×2 complex matrices, its arbitrary element can be written as

$$\widetilde{a} = \frac{1}{2}\begin{pmatrix} iq_3 & iq_1 + q_2 \\ iq_1 - q_2 & -iq_3 \end{pmatrix}. \qquad \text{(C.2.77)}$$ sl2mat

Assuming that the three complex parameters q_1, q_2, q_3 in (C.2.77) are the same as in (C.2.76), we obtain a map $so(3,\mathbb{C}) \to sl(2,\mathbb{C})$ which is obviously an isomorphism. It is straightforward to check, for example, that the commutator $[a, b]$ of any two matrices of the form (C.2.76) is mapped into the commutator $[\widetilde{a}, \widetilde{b}]$ of the corresponding matrices (C.2.77).

The symplectic algebra $sp(1,\mathbb{C})$ consists of all 2×2 matrices $\widetilde{a}$ which satisfy $\widetilde{a}\, s + s\, \widetilde{a}^T = 0$, with $s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. One can easily verify that any matrix (C.2.77) satisfies this relation, thus proving the isomorphism $sp(1,\mathbb{C}) = sl(2,\mathbb{C})$.

C.2.8 Hodge star and Excalc's

In a metric vector space, the Hodge operator establishes an map between p-forms and $(n-p)$-forms. Besides the ϑ-basis for exterior forms we can define η-basis which is Hodge dual to the ϑ-basis.

Consider an n-dimensional vector space with metric $\mathbf{g}$. Usually (when an orientation is fixed), the Hodge star is defined as a linear map $\star : \Lambda^p V^* \to \Lambda^{n-p} V^*$, such that for an arbitrary p-form ω and for an arbitrary 1-form φ it satisfies

$${}^\star(\omega \wedge \varphi) = \tilde{\mathbf{g}}^{-1}(\varphi) \rfloor {}^\star\omega \, . \qquad \text{(C.2.78)}$$ hodge2

The formula (C.2.78) reduces the definition of a Hodge dual for an arbitrary p-form to the definition of a 4-form dual to a number ${}^\star 1$. Usually, ${}^\star 1$ is taken as the ordinary (untwisted) volume form and such a procedure distinguishes a certain orientation in V. We change this convention and require instead the usual definition that

$$\star : \Lambda^p V^* \quad \longrightarrow \quad \text{twisted } \Lambda^{n-p} V^* \,, \tag{C.2.79}$$

hodge3

and vice versa

$$\star : \text{ twisted } \Lambda^p V^* \quad \longrightarrow \quad \Lambda^{n-p} V^* \,. \tag{C.2.80}$$

hodge4

Accordingly, we put $^\star 1$ equal to the *twisted* volume form:

$$^\star 1 = \eta \,. \tag{C.2.81}$$

hodge5

Let us now restrict our attention to the 4-dimensional Minkowski vector space. We can use (C.2.78) to define Hodge dual for an arbitrary p-form. Take at first $\omega = 1$ and $\varphi = \vartheta^\alpha$ as coframe 1-form. Then, with (C.2.4), (C.2.81), and (C.2.7), we find

$$^\star\vartheta^\alpha = \tilde{\mathbf{g}}^{-1}(\vartheta^\alpha) \rfloor {}^\star 1 = g^{\alpha\mu} e_\mu \rfloor \eta = \frac{1}{3!}\, \eta^\alpha{}_{\beta\gamma\delta}\, \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta \,, \tag{C.2.82}$$

hodge6

where we used (A.1.49) in order to compute the interior product $e_\mu \rfloor \eta$ for (C.2.23). We can now go on in a recursive way. Choosing in (C.2.78) $\omega = \vartheta^\alpha$ and again $\varphi = \vartheta^\beta$, we obtain:

$$^\star\!\left(\vartheta^\alpha \wedge \vartheta^\beta\right) = e^\beta \rfloor {}^\star\vartheta^\beta \,. \tag{C.2.83}$$

hodge9

Here $e^\alpha := g^{\alpha\beta}\, e_\beta$. Substituting (C.2.82) into (C.2.83) and again using (A.1.49) to evaluate the interior product, we find successively the formulas

$$^\star\vartheta^\alpha = e^\alpha \rfloor \eta = \frac{1}{3!}\, \eta^\alpha{}_{\beta\gamma\delta}\, \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta =: \eta^\alpha \,, \tag{C.2.84}$$

hodge10

$$^\star\!\left(\vartheta^\alpha \wedge \vartheta^\beta\right) = e^\beta \rfloor \left(e^\alpha \rfloor \eta\right) = \frac{1}{2!}\, \eta^{\alpha\beta}{}_{\gamma\delta}\, \vartheta^\gamma \wedge \vartheta^\delta =: \eta^{\alpha\beta} \,, \tag{C.2.85}$$

hodge11

$$^\star\!\left(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma\right) = e^\gamma \rfloor \Big(e^\beta \rfloor \left(e^\alpha \rfloor \eta\right)\Big) = \eta^{\alpha\beta\gamma}{}_\delta\, \vartheta^\delta =: \eta^{\alpha\beta\gamma} \,, \tag{C.2.86}$$

hodge12

$$^\star\!\left(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta\right) = \eta^{\alpha\beta\gamma\delta} \,. \tag{C.2.87}$$

hodge13

The newly defined η-system of p-forms, $p = 0, \ldots, 4$,

$$\left\{\eta,\, \eta^{\alpha},\, \eta^{\alpha\beta},\, \eta^{\alpha\beta\gamma},\, \eta^{\alpha\beta\gamma\delta}\right\} \qquad \text{(C.2.88)}$$ etasystem

constitutes, along with the usual ϑ-system,

$$\left\{1,\, \vartheta^{\alpha},\, \vartheta^{\alpha}\wedge\vartheta^{\beta},\, \vartheta^{\alpha}\wedge\vartheta^{\beta}\wedge\vartheta^{\gamma},\, \vartheta^{\alpha}\wedge\vartheta^{\beta}\wedge\vartheta^{\gamma}\wedge\vartheta^{\delta}\right\}, \quad \text{(C.2.89)}$$ varthetasystem

a new *metric dependent* basis for the exterior algebra over the Minkowski vector space. This construction can evidently be generalized to an n-dimensional case.

In the n-dimensional Minkowski space,

$$^{\star\star}\omega \;=\; (-1)^{p(n-p)+1}\,\omega, \quad \text{for } \omega \in \Lambda^{p}V^{*}. \qquad \text{(C.2.90)}$$ hodge14

Thus, for $n = 4$, we have $^{\star\star}\omega = \omega$ for exterior forms of *odd* degree, $p = 1, 3$, and $^{\star\star}\omega = -\omega$ for forms of *even* degree, $p = 0, 2, 4$.

From the definition (C.2.78), we can read off the rules

$$^{\star}(\phi + \psi) = {}^{\star}\phi + {}^{\star}\psi \qquad \text{and} \qquad {}^{\star}(a\,\phi) = a\,{}^{\star}\phi\,, \qquad \text{(C.2.91)}$$

for $a \in \mathbb{R}$ and $\phi,\, \psi \in \Lambda^{p}V^{*}$. These linearity properties together with (C.2.84)-(C.2.87) enable one to calculate the Hodge star of any exterior form.

The implementation of these structures in Excalc is simple. By means of the coframe statement, the metric is put in. Excalc provides the operator # as Hodge star.

In electrodynamics the most prominent role of the Hodge star operator is that it maps, up to a dimensonful factor λ, the field strength F into the excitation H, namely $H = \lambda\,{}^{\star}F$, as we will see in the fifth axiom (D.5.7). Therefore, in Excalc we simply have `excit2 := lam * # farad2;` this spacetime relation is all we need in order to make the Maxwell equations to a complete system. We used this Excalc command already in our Maxwell sample program of Sec. B.5.6.

As a further example, we study the electromagnetic energy-momentum current. We recall that we constructed in Sec. B.5.2

in Eq.(B.5.30) the electromagnetic energy-momentum tensor density in terms of the energy-momentum 3-form. The corresponding tensor we now find, instead of with $^\diamond$ rather by means of the star operator *. Thus,

```
% defs. of o(a) and maxenergy3(a) precede this declaration
pform maxenergy0(a,b)=0;
maxenergy0(-a,b):= #(o(b)^maxenergy3(-a));
```

Or, to turn to the η-system of (C.2.88). It can be programmed as follows:

```
pform eta0(a,b,c,d)=0,eta1(a,b,c)=1,
      eta2(a,b)=2,eta3(a)=3,eta4=4$

eta4           := # 1$
eta3(a)        := e(a) _|eta4$
eta2(a,b)      := e(b) _|eta3(a)$
eta1(a,b,c)    := e(c) _|eta2(a,b)$
eta0(a,b,c,d)  := e(d) _|eta1(a,b,c)$
```

We could define η alternatively as `eta4:=o(0)∧o(1)∧o(2)∧o(3)$` see (C.2.17). With these tools, the Einstein 3-form now emerges simply as

```
pform einstein3(a)=3;
einstein3(-a):=(1/2)*eta1(-a,-b,-c)^curv2(b,c);
```

Accordingly, exterior calculus and the Excalc package are really of equal power.

From a metric to the duality operator

Let us assume that a metric is introduced in a 4-dimensional vector space V. Then, the Hodge star map (C.2.79), (C.2.80) is defined for p-forms. Restricting our attention to 2-forms, we find that the Hodge star maps $M^6 = \Lambda^2 V^*$ into itself. The restriction

$$^\# = {^\star}\Big|_{\Lambda^2 V^*} \qquad \text{(C.2.92)}$$

sharphodge

is obviously a duality operator in M^6. Recalling the definitions of Sec. A.1.11, we can straightforwardly verify that both, the closure (A.1.109) and the symmetry (C.2.29), are fulfilled by (C.2.92). The matrix of the duality operator is given by

$$\#_{\rho\sigma}{}^{\alpha\beta} = \overset{g}{\kappa}_{\rho\sigma}{}^{\alpha\beta} := \eta_{\mu\nu\rho\sigma}\, g^{\alpha[\mu} g^{\nu]\beta}. \tag{C.2.93}$$

sharp3

Alternatively, in 6-dimensional notation, it reads

$$\#_I{}^J = \overset{g}{\kappa}_I{}^J := \begin{pmatrix} \overset{g}{C} & \overset{g}{A} \\ \overset{g}{B} & \overset{g}{C}{}^{\mathrm{T}} \end{pmatrix}, \tag{C.2.94}$$

sparpIJ0

where a straightforward calculation yields the 3×3 blocks

$$\overset{g}{A}{}^{ab} = \sqrt{-g}\,\left(g^{00} g^{ab} - g^{0a} g^{0b}\right), \tag{C.2.95}$$

Ahodge

$$\overset{g}{B}{}_{ab} = \frac{1}{4}\sqrt{-g}\,\left(g^{ce} g^{df} - g^{de} g^{ef}\right) \hat{\epsilon}_{acd}\, \hat{\epsilon}_{bef}, \tag{C.2.96}$$

Bhodge

$$\overset{g}{C}{}^{a}{}_{b} = \frac{\sqrt{-g}}{2}\left(g^{0c} g^{ad} - g^{ac} g^{0d}\right) \epsilon_{bcd}\,. \tag{C.2.97}$$

Chodge

C.2.9 Manifold with a metric, Levi-Civita connection

The metric on a manifold is introduced pointwise as a smooth scalar product on the tangent spaces. The Levi-Civita (or Riemannian) connection is a unique linear connection with vanishing torsion and covariantly constant metric.

Let X_n be an n-dimensional differentiable manifold. We say that a *metric* is defined on X_n, if a metric tensor $\mathbf{g}$ is smoothly assigned to the tangent vector spaces X_x at each point x. In terms of the coframe field,

$$\mathbf{g} = g_{\alpha\beta}(x)\, \vartheta^\alpha \otimes \vartheta^\beta, \quad \text{where} \quad g_{\alpha\beta}(x) = \mathbf{g}(e_\alpha, e_\beta) \tag{C.2.98}$$

metlocal

is a smooth tensor field in every local coordinate chart. The manifold with a metric structure defined on it is called a *(pseudo)-Riemannian* manifold, denoted $V_n = (X_n, \mathbf{g})$. Usually, a metric (and, correspondingly, a manifold) is called Riemannian, if the $g_{\alpha\beta}(x)$ is positive definite for all x. However, in order to simplify formulations, we will omit the 'pseudo' and call Riemannian also metrics with a Lorentzian signature.

The metric brings a whole bunch of related objects on a manifold V_n. First of all, a metric volume n-form emerges on a V_n: For every coframe field $\vartheta^\alpha = (\vartheta^{\hat{1}}, \ldots, \vartheta^{\hat{n}})$ it is defined by

$$\eta := \sqrt{-\det g_{\mu\nu}}\,\vartheta^{\hat{1}} \wedge \cdots \wedge \vartheta^{\hat{n}} = \frac{1}{n!}\,\eta_{\alpha_1\ldots\alpha_n}\,\vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_n} \tag{C.2.99}$$ etaN

$$= \sqrt{-\det g_{kl}}\,dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n!}\,\eta_{i_1\ldots i_n}\,dx^{i_1} \wedge \cdots \wedge dx^{i_n}. \tag{C.2.100}$$ etaNdx

The *world metric tensor* with components

$$g_{ij}(x) = e_i{}^\alpha(x)\,e_j{}^\beta(x)\,g_{\alpha\beta}(x) \tag{C.2.101}$$ metworld

is defined in every local coordinate chart $\{x^i\}$. The principal difference between $g_{\alpha\beta}$ and g_{ij} is that the former always can be 'gauged away' by the suitable choice of the frame field e_α. One can choose an orthonormal frame field, e.g., in which $g_{\alpha\beta}$ has the diagonal form (C.2.8) independent of local coordinates. However, it is impossible in general to choose the coordinates $\{x^i\}$ in such a way that g_{ij} is constant everywhere on the V_n.

The Levi-Civita tensor densities in (C.2.99), (C.2.100) are introduced by

$$\begin{aligned}\eta_{\alpha_1\ldots\alpha_n}(x) &= \sqrt{-\det g_{\mu\nu}}\,\epsilon_{\alpha_1\ldots\alpha_n},\\ \eta_{i_1\ldots i_n}(x) &= e_{i_1}{}^{\alpha_1}\ldots e_{i_n}{}^{\alpha_n}\,\eta_{\alpha_1\ldots\alpha_n} = \sqrt{-\det g_{kl}}\,\epsilon_{i_1\ldots i_n},\end{aligned} \tag{C.2.102}$$

with the numerical permutation symbol chosen as $\epsilon_{1\ldots n} = +1$. Hats over numerical indices help to distinguish components with respect to local frames form components with respect to coordinate frames.

The next relative of the metric is the Hodge star operator. It is naturally introduced on a V_n pointwise with the help of formulas derived in Sec. C.2.8.

Finally, the most far-reaching and non-trivial consequence of the metric $\mathbf{g}$ is the existence on a V_n of a special covariant differentiation $\widetilde{\nabla}$ which is usually called a *Riemannian* or *Levi-Civita connection*. We will use tilde to denote this connection and any objects or operators constructed from it. As was shown in Sec. C.1.1, a covariant differentiation is defined on a manifold as soon as in every local chart the connection 1-forms $\Gamma_i{}^j$ are given which obey the consistency condition (C.1.11). The Riemannian connection is defined, in each local coordinate system, by the *Christoffel symbols* $\widetilde{\Gamma}_{ki}{}^j$:

$$\widetilde{\Gamma}_i{}^j \;=\; \widetilde{\Gamma}_{ki}{}^j\, dx^k, \qquad \widetilde{\Gamma}_{ki}{}^j \;:=\; \frac{1}{2}\, g^{jl}\left(\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}\right). \tag{C.2.103}$$

Chris

The Levi-Civita (or Riemannian) connection $\widetilde{\nabla}$ has some special properties which distinguish it from other covariant differentiations. It has *vanishing torsion*. This is trivially seen from (C.1.43) and (C.1.18) if we notice that the Christoffel symbols (C.2.103) are symmetric in its lower indices. Moreover, the covariant exterior derivative of the metric with respect to the Levi-Civita connection vanishes identically:

$$\widetilde{D} g_{\alpha\beta} = d g_{\alpha\beta} - \widetilde{\Gamma}_\beta{}^\lambda\, g_{\alpha\lambda} - \widetilde{\Gamma}_\alpha{}^\lambda\, g_{\lambda\beta} = d g_{\alpha\beta} - 2\, \widetilde{\Gamma}_{(\alpha\beta)} = 0, \tag{C.2.104}$$

zeroDg

see (C.2.134). A connection for which the covariant derivative of the metric is zero, is called *metric-compatible.*

C.2.10 Codifferential and wave operator, also in Excalc

By means of the Hodge star operator, we can define the codifferential which is adjoint to the exterior differential d with respect to the scalar product on exterior forms. This yields directly a wave operator.

Consider the space $\Lambda^p(X)$ of all smooth exterior p-forms on X. The Hodge operator makes it possible to define a natural scalar product on this functional space:

$$(\omega,\varphi) := \int\limits_X \omega\wedge{}^\star\varphi, \qquad \text{for all} \qquad \omega,\varphi\in\Lambda^p(X). \tag{C.2.105}$$

p-scal

Then the *codifferential* operator $d^\dagger$ can be introduced as an adjoint to the exterior differential d with respect to the scalar product (C.2.105),

$$(\omega, d^\dagger\varphi) := (d\omega,\varphi). \tag{C.2.106}$$

By construction, the codifferential maps p-forms into $(p-1)$-forms (contrary to the exterior differential which increases the rank of a form by one). Using the property (C.2.90) of the Hodge operator, one can verify that in an n-dimensional Lorentzian space the *codifferential on p-forms* is given explicitly by

$$d^\dagger = (-1)^{n(p-1)}\,{}^\star d\,{}^\star. \tag{C.2.107}$$

ddagger

The Leibniz rule for d was used at an intermediate step, and the boundary integral is vanishing due to the proper behavior of the forms at infinity. Accordingly, in $n=4$ we have $d^\dagger = {}^\star d\,{}^\star$ for all forms.

In local coordinates, the codifferential of an arbitrary p-form $\varphi = \frac{1}{p!}\,\varphi_{i_1\dots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ reads:

$$d^\dagger\varphi = \frac{-1}{(p-1)!}\widetilde{\nabla}^j\varphi_{ji_1\dots i_{p-1}}\,dx^{i_1}\wedge\cdots\wedge dx^{i_{p-1}}. \tag{C.2.108}$$

ddaggerloc

Here $\widetilde{\nabla}$ is the covariant differentiation for the Levi-Civita connection (C.2.103).

The codifferential is nilpotent

$$d^\dagger d^\dagger = 0 \tag{C.2.109}$$

which follows directly from (C.2.107), (C.2.90) and the nilpotency property $d^2=0$ of the exterior differential (A.2.19). The operator d (resp., $d^\dagger$) increases (resp., decreases) the rank of a form by 1. Hence, the combinations $dd^\dagger$ and $d^\dagger d$ both map p-forms into p-forms. However, these operators are not self-adjoint with respect to the scalar product (C.2.105). The symmetrized second order differential operator

$$\Box := d^\dagger\, d + d\, d^\dagger \qquad \text{(C.2.110)}$$

d'Alembertian

is called the wave operator (or d'Alembertian, also called Laplace-Beltrami operator). It is, by construction, self-adjoint with respect to (C.2.105).

On one occasion, we had to check whether the wave operator, if applied to a certain coframe field, vanishes, i.e., $\Box\vartheta^a \stackrel{?}{=} 0$. Excalc could help. After the `coframe` statement specifying the appropriate value for the coframe, we defined a suitable 1-form:

```
pform wavetocoframe1(a)=1$

wavetocoframe1(a):= d(#(d(#o(a)))) + #(d(#(d o(a))));
```

The emerging expression we had to treat with switches and suitable substitutions, but the quite messy computation of the wave operator was given to the machine.

C.2.11 ⊗Nonmetricity

Let a connection and a metric be defined independently on the same spacetime manifold. Then the nonmetricity is a measure of the incompatibility between metric and connection.

Let us consider the general case when on a manifold X_n metric and connection are defined independently. Such a manifold is denoted $(X_n, \nabla, \mathbf{g})$ and is called a *metric-affine* spacetime. Since the metric is a tensor field of type $\left[{0 \atop 2}\right]$, its covariant differentiation yields a type $\left[{0 \atop 3}\right]$ tensor field which is called *nonmetricity*:

$$Q(u,v,w) := \mathbf{g}(\nabla_u v, w) + \mathbf{g}(v, \nabla_u w) - u\{\mathbf{g}(v,w)\}\,, \qquad \text{(C.2.111)}$$

nonmeticity

for all vector fields u, v, w. Nonmetricity measures the extent to which a connection ∇ is incompatible with the metric $\mathbf{g}$. Metric-compatibility (also called metricity) $Q(u,v,w) = 0$ implies the conservation of lengths and angles under parallel transport. A manifold which is endowed with a metric and a metric-compatible connection is said to be a *Riemann–Cartan* manifold (or a U_n). In general, the Riemann-Cartan manifold has a non-vanishing torsion. When the latter is zero, we recover the Riemannian manifold described in Sec. C.2.9.

Similarly to the 2-forms of torsion (C.1.39), (C.1.41) and curvature (C.1.46), (C.1.47), we define the nonmetricity 1-form

$$Q_{\alpha\beta} = Q_{\gamma\alpha\beta}\,\vartheta^{\gamma} \qquad \text{(C.2.112)}$$

by

$$Q_{\gamma\alpha\beta} = Q_{\alpha\beta}(e_\gamma) = Q(e_\gamma, e_\alpha, e_\beta)\,. \qquad \text{(C.2.113)}$$ nonexp

Since $u(g_{\alpha\beta}) = dg_{\alpha\beta}\,(u)$, equation (C.2.111) is equivalent to

$$Q_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma_{\alpha\beta} + \Gamma_{\beta\alpha} = -Dg_{\alpha\beta}\,, \qquad \text{(C.2.114)}$$ structure0

where $\Gamma_{\alpha\beta} := g_{\beta\gamma}\Gamma_{\alpha}{}^{\gamma}$. If the $g_{\alpha\beta}$ are constants, then it follows from (C.2.114) that $Q_{\alpha\beta} = 2\,\Gamma_{(\alpha\beta)}$. Hence, in a U_n, where $Q_{\alpha\beta} = 0$, we have antisymmetric connection one-forms

$$\Gamma_{\alpha\beta} \overset{*}{=} -\Gamma_{\beta\alpha}\,, \qquad \text{(C.2.115)}$$

provided the $g_{\alpha\beta}$ are constants, i.e., with respect to orthonormal coframe fields, e.g.

We shall refer to (C.2.114) as the *0th Cartan structural relation* and shall call the expression obtained as its exterior derivative,

$$DQ_{\alpha\beta} = dQ_{\alpha\beta} - \Gamma_{\alpha}{}^{\gamma}\wedge Q_{\gamma\beta} - \Gamma_{\beta}{}^{\gamma}\wedge Q_{\alpha\gamma} = 2\,R_{(\alpha\beta)}\,, \qquad \text{(C.2.116)}$$ bianchi0

the *0th Bianchi identity.* The proof of (C.2.116) makes use of the Ricci identity (C.1.66).

It is convenient to separate the trace part of the nonmetricity from its traceless piece. Let us define a *Weyl 1-form (or Weyl covector)* by

$$Q := \frac{1}{n}\, Q_{\alpha\beta}\, g^{\alpha\beta}. \qquad \text{(C.2.117)}$$

weyl1form

The factor $1/n$ is conventional. Then the nonmetricity is decomposed into its *deviator* $\not{Q}_{\alpha\beta}$ and its trace according to

$$Q_{\alpha\beta} = \not{Q}_{\alpha\beta} + Q\, g_{\alpha\beta}\,. \qquad \text{(C.2.118)}$$

QQ

The trace of the curvature, which is called the *segmental curvature*, can be expressed in terms of the Weyl 1-form:

$$R_{\gamma}{}^{\gamma} = \frac{n}{2}\, dQ. \qquad \text{(C.2.119)}$$

The physical importance of the Weyl 1-form is related to the fact that, during parallel transport, the contribution of the Weyl 1-form does *not* affect the lightcone, whereas lengths of non-null vectors are merely scaled with some (path-dependent) factor. A space with $\not{Q}_{\alpha\beta} = 0$ is called a *Weyl-Cartan* space Y_n. In this latter case, the position vector (C.1.61) changes according to

$$\Delta r^{\alpha} = \int_{S} (T^{\alpha} - \frac{1}{n} R_{\gamma}{}^{\gamma}\, r^{\alpha} \, - R^{[\beta\alpha]}\, r_{\beta}), \qquad \text{(C.2.120)}$$

cartandilat

if it is Cartan displaced over a closed loop which encircles S. The first curvature term induces a dilation, while the second one is a pure rotation.

C.2.12 ⊗Post-Riemannian pieces of the connection

An arbitrary linear connection can always be split into the Levi-Civita connection plus a post-Riemannian tensorial piece called the distortion. The latter depends on torsion and nonmetricity. Correspondingly, all the geometric objects and operators can be systematically decomposed into Riemannian and post-Riemannian parts.

"...the question whether this [spacetime] continuum is Euclidean or structured according to the Riemannian scheme or still otherwise is a genuine physical question which has to be answered by experience rather than being a mere convention to be chosen on the basis of expediency."[3]

The geometrical properties of an arbitrary metric-affine spacetime are described by the 2-forms of curvature $R_\alpha{}^\beta$ and torsion T^α and by the 1-form of nonmetricity $Q_{\alpha\beta}$. Particular values of these fundamental objects specify different geometries which may be realized on a spacetime manifold. Physically, one can think of a number of 'phase transitions' which the spacetime geometry undergoes at different energies (or distance scales). Correspondingly, it is convenient to study particular realizations of geometrical structures within the framework of several specific gravitational models. The overview of these models and of the relevant geometries is given in Fig. C.2.2.

The most general gravitational model – metric-affine gravity (MAG) – employs the (L_n, g) geometry in which all three main objects, curvature, torsion, and nonmetricity are non-trivial. Such a geometry could be realized at extremely small distances (high energies) when the hypermomentum current of matter fields plays a central role.

Other gravitational models and the relevant geometries appear as special cases when one or several main geometrical objects are completely or partly "switched off". The Z_4 geometry is characterized by $T^\alpha = 0$ and was used in the unified field theory of Eddington and in so-called SKY-gravity (theories of Stephenson-Kilmister-Yang). Switching off the traceless nonmetricity, $\not{Q}_{\alpha\beta} = 0$, yields the Weyl-Cartan space Y_4 (with torsion) or standard Weyl theory W_4 (with $T^\alpha = 0$).

Furthermore, switching off the nonmetricity completely, one recovers the Riemann-Cartan geometry U_4 which is the arena of Poincaré gauge (PG) gravity in which the spin current of matter, besides its energy-momentum current, is an additional source of the gravitational field. The Riemannian geometry V_4 (with

[3] A. Einstein: *Geometrie und Erfahrung* [1], our translation.

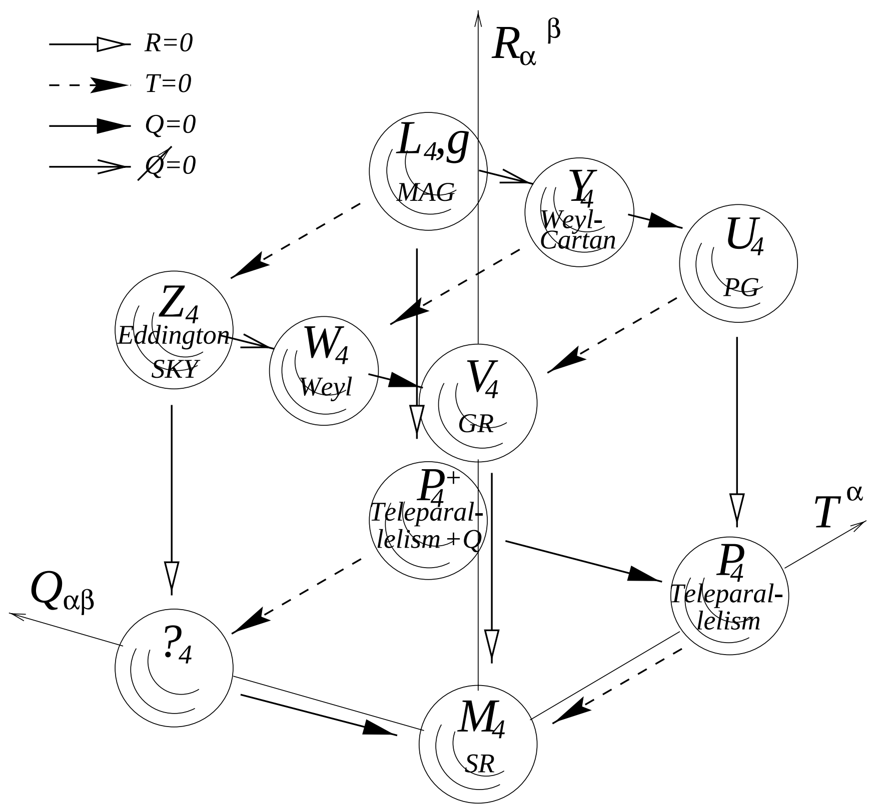

Figure C.2.2: MAGic cube: Classification of geometries and gravity theories in the three "dimensions" (R, T, Q).

$Q_{\alpha\beta} = 0$ and $T^\alpha = 0$) describes, via Einstein's General Relativity (GR), gravitational effects on a macroscopic scale when the energy-momentum current is the only source of gravity. Finally, when the curvature is zero, $R_\alpha{}^\beta = 0$, one obtains the Weitzenböck space P_4 and the *teleparallelism* theory of gravity (when $Q_{\alpha\beta} = 0$) or a generalized teleparallelism theory in the spacetime with nontrivial torsion and nonmetricity P_4^+. The Minkowski spacetime M_4 with vanishing $Q_{\alpha\beta} = 0$, $T^\alpha = 0$, $R_\alpha{}^\beta = 0$ underlies Special Relativity (SR) theory. The relations between the different theories and geometries are given in Fig. C.2.2 by means of arrows of different type which specify which object is switched off.

In a metric-affine space, curvature, torsion, and nonmetricity satisfy the three Bianchi identities (C.2.116), (C.1.68), and (C.1.70):

$$DQ_{\alpha\beta} = 2R_{(\alpha\beta)}, \qquad \text{0th Bianchi identity} \qquad \text{(C.2.121)}$$ Bia0

$$DT^\alpha = R_\beta{}^\alpha \wedge \vartheta^\beta, \qquad \text{1st Bianchi identity} \qquad \text{(C.2.122)}$$ Bia1

$$DR_\alpha{}^\beta = 0. \qquad \text{2nd Bianchi identity} \qquad \text{(C.2.123)}$$ Bia2

In practical calculations, it is important to know exactly the number of geometrical and physical variables and their algebraic properties (e.g., symmetries, orthogonality relations, etc.). These aspects can be clarified with the help of two types of decompositions. A linear connection can always be decomposed into Riemannian and post-Riemannian parts,

$$\Gamma_\beta{}^\alpha = \widetilde{\Gamma}_\beta{}^\alpha + N_\beta{}^\alpha, \qquad \text{(C.2.124)}$$ decom

where the *distortion* 1-form $N_{\alpha\beta}$ is expressed in terms of torsion and nonmetricity as follows:

$$N_{\alpha\beta} = -e_{[\alpha} \rfloor T_{\beta]} + \frac{1}{2}(e_\alpha \rfloor e_\beta \rfloor T_\gamma)\, \vartheta^\gamma + (e_{[\alpha} \rfloor Q_{\beta]\gamma})\, \vartheta^\gamma + \frac{1}{2} Q_{\alpha\beta}. \qquad \text{(C.2.125)}$$ N

The distortion "measures" a deviation of a particular geometry from the purely Riemannian one. As a by-product of the decomposition (C.2.124) we verify that a metric-compatible connection without torsion is unique: it is the Levi-Civita connection. Nonmetricity and torsion can easily be recovered from the distortion, namely

$$Q_{\alpha\beta} = 2\, N_{(\alpha\beta)}\,, \qquad T^{\alpha} = N_{\beta}{}^{\alpha} \wedge \vartheta^{\beta}\,. \tag{C.2.126}$$ distorsion2

If we collect then the information we have on the splitting of a connection into Riemannian and non-Riemannian pieces, then, in terms of the metric $g_{\alpha\beta}$, the coframe ϑ^{α}, the anholonomity C^{α}, the torsion T^{α}, and the nonmetricity $Q_{\alpha\beta}$, we have the highly symmetric master formula

$$\begin{aligned}\Gamma_{\alpha\beta} = \frac{1}{2}dg_{\alpha\beta} + (e_{[\alpha} \rfloor dg_{\beta]\gamma})\vartheta^{\gamma} + e_{[\alpha} \rfloor C_{\beta]} - \frac{1}{2}(e_{\alpha} \rfloor e_{\beta} \rfloor C_{\gamma})\vartheta^{\gamma} \quad &(V_n)\\ - e_{[\alpha} \rfloor T_{\beta]} + \frac{1}{2}(e_{\alpha} \rfloor e_{\beta} \rfloor T_{\gamma})\vartheta^{\gamma} \quad &(U_n)\\ + \frac{1}{2}Q_{\alpha\beta} + (e_{[\alpha} \rfloor Q_{\beta]\gamma})\vartheta^{\gamma}\,. \quad &(L_n, g)\,.\end{aligned} \tag{C.2.127}$$ masternonriem

The first line of this formula refers to the Riemannian part of the connection; together with the second line a Riemann-Cartan geometry is encompassed; and only the third line makes the connection a really independent quantity. Note that locally the torsion T_{α} can be mimicked by the negative of the anholonomity $-C_{\alpha}$ and the nonmetricity $Q_{\alpha\beta}$ by the exterior derivative $dg_{\alpha\beta}$ of the metric.

In a 4-dimensional metric-affine space, the curvature has 96 components, torsion 24, and nonmetricity 40. In order to make the work with all these variable manageable, one usually decomposes all geometrical quantities in irreducible pieces with respect to the Lorentz group.

C.2.13 Excalc again

Excalc has a commodity: If a `coframe o(a)` and a `metric g` are prescribed, it calculates the Riemannian piece of the connection $\widetilde{\Gamma}_{\alpha}{}^{\beta}$ on demand. One just has to issue the command `riemannconx chris1;` then chris1 (one could take any other name) is, without further declaration, a 1-form with the index structure `chris1(a,b)`. Since we use Schouten's conventions in this book, we have to redefine Schrüfer's riemannconx according to `chris1(a,-b):=chris1(-b,a);`

If you distrust Excalc, you could also compute your own Riemannian connection according to (C.2.127), i.e.,

```
pform anhol2(a)=2,christ1(a,b)=1$

anhol2(a)      := d o(a)$
christ1(-a,-b):= (1/2)*d g(-a,-b)
   +(1/2)*((e(-a)_|(d g(-b,-c)))-(e(-b)_|(d g(-a,-c))))^o(c)
   +(1/2)*( e(-a)_|anhol2(-b)   - e(-b)_|anhol2(-a))
   -(1/2)*( e(-a)_|(e(-b)_|anhol2(-c)))^o(c)$
```

But I can assure you that Excalc does its job correctly.

In any case, with `chris1(a,b)` or with `christ1(a,b)` you can equally well compute the distortion 1-form and the nonmetricity 1-form in terms of coframe `o(a)`, metric `g`, and connection `conn1(a,b)`. The calculation would run as follows:

```
coframe o(0)=...;            % input 1
frame e;
riemannconx chris1;  chris1(a,b):=chris1(b,a);
pform conn1(a,b)=1, distor1(a,b)=1, ,nonmet1(a,b)=1$

conn1(0,0):=...;             % input 2
distor1(a,b):=conn1(a,b)-chris1(a,b);
nonmet1(a,b):=distor1(a,b)-distor1(b,a);
```

and one could continue in this line and compute torsion and curvature according to

```
pform torsion2(a)=2, curv2(a,b)=2;

torsion2(a):=d o(a)+conn1(-b,a)^o(b);
curv2(-a,b):=d conn1(-a,b)-conn1(-a,c)^conn1(-c,b);
```

All other relevant geoemtrical quantities can be derived thereform.

Problems

Problem C.1.

Check the geometrical interpretation of torsion given in Fig. C.1.1 by direct calculation using the definition of the parallel transport.

Problem C.2.

Prove (C.1.53) in the infinitesimal case, approximating the curve σ by a small parallelogram.

Problem C.3.

Prove the following relations involving the transposed connection:

1. $T^\alpha = -\overset{\frown}{T}{}^\alpha$, i.e. $T^\alpha = -\overset{\frown}{D}\,\vartheta^\alpha$;

2. $\overset{\frown}{D}\,\epsilon^{\alpha\beta\gamma\delta} = 0$;

3. $\text{Ł}_{e_\alpha}\vartheta^\beta = e_\alpha \rfloor T^\beta$;

4. The covariant Lie derivative of an arbitrary p-form $\Psi = \Psi_{\alpha_1\dots\alpha_p}\,\vartheta^{\alpha_1}\wedge\cdots\wedge\vartheta^{\alpha_p}/p!$:

$$\text{Ł}_{e_\alpha}\Psi = \frac{1}{p!}\left(\overset{\frown}{D}_\alpha \Psi_{\alpha_1\dots\alpha_p}\right)\vartheta^{\alpha_1}\wedge\cdots\wedge\vartheta^{\alpha_p}, \qquad \text{(C.2.128)}$$ covarLIE

Problem C.4.

1. Find a transformation matrix from an orthonormal basis to the half-null frame in which metric has the form (C.2.11).

2. Find a transformation matrix from an orthonormal basis to Newman-Penrose null frame in which metric has the form (C.2.13).

3. Find a linear transformation $e_\alpha = L_\alpha{}^\beta\, f_\beta$ which brings the Finkelstein basis f_β back to an orthonormal frame e_α.

Solution:

$$L_\alpha{}^\beta = \frac{1}{2}\begin{pmatrix} 1/\sqrt{3} & \sqrt{2} & 0 & 1 \\ 1/\sqrt{3} & 0 & \sqrt{2} & -1 \\ 1/\sqrt{3} & 0 & -\sqrt{2} & -1 \\ 1/\sqrt{3} & -\sqrt{2} & 0 & 1 \end{pmatrix}. \qquad \text{(C.2.129)} \quad \texttt{ttfnn}$$

Check that (C.2.14) is inverse to (C.2.129), up to a Lorentz transformation.

4. Show that the matrix (C.2.129) is represented as a product $L = S\,R$ of the two matrices:

$$S = \begin{pmatrix} -\sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \qquad \text{(C.2.130)}$$

The matrix S just scales the time coordinate and a short calculation with REDUCE shows that R is an element of $SO(4)$.

Problem C.5.

1. Prove that for $\phi,\ \psi \in \Lambda^p V^*$

$$^\star\phi \wedge \psi = {}^\star\psi \wedge \phi. \qquad \text{(C.2.131)} \quad \texttt{moveast}$$

2. If $\phi \in \Lambda^p V^*$, show that

$$\vartheta^\alpha \wedge (e_\alpha \rfloor \phi) = p\,\phi\,, \qquad \text{(C.2.132)} \quad \texttt{ephi}$$

$$^\star(\phi \wedge \vartheta_\alpha) = e_\alpha \rfloor {}^\star\phi\,. \qquad \text{(C.2.133)} \quad \texttt{east}$$

3. Show that in 4-dimensional space

   (a) $\vartheta^\alpha \wedge \eta_\beta = \delta^\alpha_\beta\ \eta\,,$

   (b) $\vartheta^\alpha \wedge \eta_{\beta\gamma} = \delta^\alpha_\gamma\ \eta_\beta - \delta^\alpha_\beta\ \eta_\gamma\,,$

   (c) $\vartheta^\alpha \wedge \eta_{\beta\gamma\delta} = \delta^\alpha_\delta\ \eta_{\beta\gamma} + \delta^\alpha_\gamma\ \eta_{\delta\beta} + \delta^\alpha_\beta\ \eta_{\gamma\delta}\,,$

   (d) $\vartheta^\alpha\ \eta_{\beta\gamma\delta\mu} = \delta^\alpha_\mu\ \eta_{\beta\gamma\delta} - \delta^\alpha_\delta\ \eta_{\beta\gamma\mu} + \delta^\alpha_\gamma\ \eta_{\beta\delta\mu} - \delta^\alpha_\beta\ \eta_{\gamma\delta\mu}\,.$

   Hint: $\vartheta^\alpha \wedge \eta = 0$ because it is a 5-form in a 4-dimensional space.

Problem C.6.

1. Prove that Christoffel symbols define a covariant differentiation by checking the transformation law (C.1.11) for the one-forms (C.2.103).

2. Show that with respect to an arbitrary local frame field, the Riemannian connection form reads, cf. (C.1.18):

$$
\begin{aligned}
\widetilde{\Gamma}_\alpha{}^\beta &= e_j{}^\beta \, \widetilde{\Gamma}_i{}^j \, e^i{}_\alpha + e_i{}^\beta \, d\, e^i_\alpha \\
&= \frac{1}{2}\, g^{\beta\gamma} \Big(dg_{\alpha\gamma} + (e_\alpha \rfloor dg_{\gamma\lambda} - e_\gamma \rfloor dg_{\alpha\lambda})\, \vartheta^\lambda \Big) \\
&\quad + \frac{1}{2} \Big(e_\alpha \rfloor C^\beta \; - e^\beta \rfloor C_\alpha \; - (e_\alpha \rfloor e^\beta \rfloor C_\gamma)\, \vartheta^\gamma \Big),
\end{aligned}
\tag{C.2.134}
$$

nonChris

where $C^\alpha = d\vartheta^\alpha$ is the anholonomity 2-form, see (A.2.35).

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References Part C

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Part D

The Maxwell-Lorentz spacetime relation

So far, the Maxwell equations (B.4.8) and (B.4.9) represent an underdetermined system of partial differential equations of first order for the excitation H and the field strength F. In order to reduce the number of independent variables, we have to set up a relation between H and F,

$$H = H(F)\,.$$

We will call this the *electromagnetic spacetime relation.* Therefore we can complete electrodynamics, formulated in Part B up to now metric- and connection-free, by introducing a suitable spacetime relation as *fifth axiom.*

The simplest choice is, of course, a *linear* relation $H \sim \kappa \circ F$, with the "constitutive" tensor κ. This yields eventually conventional Maxwell-Lorentz electrodynamics. It is remarkable that this linear relation, if supplemented merely by a *closure* property (basically $\kappa \circ \kappa \sim -1$) and a *symmetry* of κ (namely $\kappa \sim \kappa^{\mathrm{T}}$), induces a *lightcone* at each point of spacetime. In other words, we are able to *derive* the conformally invariant part of the *metric* of spacetime from the existence of a linear κ together with its closure property and its symmetry.[4]

Alternatively, one could simply assume the existence of a (pseudo-) Riemannian metric $\mathbf{g}$ of signature $(+1, -1, -1, -1)$ on the spacetime manifold. In both cases, the Hodge star operator $\star$ is available and ordinary electrodynamics can be recovered via the spacetime relation $H \sim {}^\star F$.

---

[4]This type of ideas goes back to Toupin [35] and Schönberg [30], see also Urbantke [37] and Jadczyk [12]. Wang [38] gave a revised presentation of Toupin's results. A forerunner was Peres [23], see in this context also the more recent papers by Piron and Moore [26]. For new and recent results, see [20]. It was recognized by Brans [1] that, within general relativity, it is possible to define a duality operator in much the same way as we will present it below, see (D.3.11), and that from this duality operator the metric can be recovered. Subsequently numerous authors discussed such structures in the framework of general relativity theory, see, e.g., Capovilla, Jacobson, and Dell [2], 't Hooft [11], Harnett [8, 9], Obukhov and Tertychniy [21], and the references given there. In the present Part D, we will also use freely the results of Gross and Rubilar [6, 28].

D.1

Linearity between H and F and quartic wave surface

We assume a linear spacetime relation between excitation H and field strength F encompassing 36 independent components of the constitutive tensor. As a consequence, the wave vectors of electromagnetic vacuum waves lie on quartic surfaces. This is unphysical and requires additional constraints.

D.1.1 Linearity

The *electromagnetic spacetime relation*[1] expreses the excitation H in terms of the field strength F. Both are elements of the space of 2-forms $\Lambda^2 X$. However, H is twisted and F is unwisted. Thus one can formulate the spacetime relation as

$$H = \boldsymbol{\kappa}(F)\,, \qquad \text{(D.1.1)}$$ oper1

where

$$\boldsymbol{\kappa} : \; \Lambda^2 X \;\to\; \Lambda^2 X \qquad \text{(D.1.2)}$$ oper2

[1]Post [27] named it *constitutive map* including also the constitutive relation for matter, see (E.3.16), Truesdell & Toupin [36], Toupin [35], and Kovetz [15] use the term *aether relations*.

is an *invertible* operator that maps an untwisted 2-form in a twisted 2-form and vice versa.

The most important case is that of a *linear* law between the 2-forms H and F. Accordingly, the operator (D.1.2) is required to be linear, i.e., for all $a, b \in \Lambda^0 X$ and $\phi, \psi \in \Lambda^2 X$ we have

$$\boldsymbol{\kappa}(a\,\phi + b\,\psi) = a\,\boldsymbol{\kappa}(\phi) + b\,\boldsymbol{\kappa}(\psi)\,. \tag{D.1.3}$$ linear0

For physical applications, it may be useful to present our linear operator $\boldsymbol{\kappa}$ in a more explicit form. Because of its linearity, it is sufficient to know the action of $\boldsymbol{\kappa}$ on the basis 2-forms. The corresponding mathematical preliminaries were outlined in Sec. A.1.10. A choice of the natural coframe $\vartheta^i = dx^i$ yields the specific 2-form basis B^I of (A.1.81). The operator $\boldsymbol{\kappa}$ acts on the 2-form basis $dx^k \wedge dx^l$ ($= B^I$ in the equivalent bivector language) and maps them in twisted 2-forms the latter of which we can again decompose:

$$\boldsymbol{\kappa}\left(dx^k \wedge dx^l\right) = \frac{1}{2}\,\kappa_{ij}{}^{kl}\,dx^i \wedge dx^j \quad \text{or} \quad \boldsymbol{\kappa}\left(B^K\right) = \kappa_I{}^K\,B^I\,. \tag{D.1.4}$$ oper5

Now, we decompose the 2-forms in (D.1.1):

$$H = \frac{1}{2}\,H_{ij}\,dx^i \wedge dx^j \quad \text{or} \quad H = H_I\,B^I\,. \tag{D.1.5}$$ oper3

Substituting (D.1.5), together with the similar expansion for F, and making use of (D.1.4), we find

$$H_{ij} = \frac{1}{2}\,\kappa_{ij}{}^{kl}\,F_{kl} \quad \text{or} \quad H_I = \kappa_I{}^K\,F_K\,. \tag{D.1.6}$$ chiHF

Thus a linear spacetime relation postulates the existence of 6×6 functions $\kappa_{ij}{}^{kl}$ with

$$\kappa_{ij}{}^{kl} = -\,\kappa_{ji}{}^{kl} = -\,\kappa_{ij}{}^{lk}\,. \tag{D.1.7}$$ oper6

In the bivector notation, these 36 independent components are arranged into a 6×6 matrix $\kappa_I{}^K$.

The choice of the local coordinates is unimportant. In a different coordinate system, the linear law preserves its form due to the tensorial transformation properties of ${\kappa_{ij}}^{kl}$. Alternatively, instead of the local coordinates, one may choose an arbitrary (anholonomic) coframe $\vartheta^\alpha = {e_i}^\alpha dx^i$ and may then decompose the two-forms H and F with respect to it according to $H = H_{\alpha\beta}\,\vartheta^\alpha \wedge \vartheta^\beta/2$ and $F = F_{\alpha\beta}\,\vartheta^\alpha \wedge \vartheta^\beta/2$. Then, if we redo the calculations of above, we find

$$H_{\alpha\beta} = \frac{1}{2}\,{\kappa_{\alpha\beta}}^{\gamma\delta}\,F_{\gamma\delta} \quad \text{with} \quad {\kappa_{\alpha\beta}}^{\gamma\delta} = {e^i}_\alpha\,{e^j}_\beta\,{e_k}^\gamma {e_l}^\delta\,{\kappa_{ij}}^{kl}\,. \tag{D.1.8}$$

chiHFanh

Here we used also the components of the frame $e_\alpha = {e^k}_\alpha\,\partial_k$.

As we recall from Sec. A.1.10, the Levi-Civita symbols (A.1.91) and (A.1.93) can be used as a "metric" for raising and lowering (pairs of) indices. We define

$$\chi^{IK} := \epsilon^{IM}\,{\kappa_M}^K \quad \text{or} \quad \chi^{ijkl} = \frac{1}{2}\,\epsilon^{ijmn}\,{\kappa_{mn}}^{kl} \tag{D.1.9}$$

raise

and, conversely,

$${\kappa_I}^K = \hat{\epsilon}_{IM}\,\chi^{MK} \quad \text{or} \quad {\kappa_{ij}}^{kl} = \frac{1}{2}\,\hat{\epsilon}_{ijmn}\,\chi^{mnkl}\,. \tag{D.1.10}$$

lower

The 36 functions ${\kappa_{ij}}^{kl}(\sigma, x)$ as well as the $\chi^{ijkl}(\sigma, x)$ depend on time σ and on space x in general. Because of the corresponding properties of the Levi-Civita symbol, the χ^{ijkl} represent an *(untwisted) tensor density of weight* $+1$.

Excitation H and field strength F are measurable quantities. The functions ${\kappa_{ij}}^{kl}$ (or χ^{ijkl}) are "quotients" of H and F. Thus they first of all carry the dimension $[\boldsymbol{\kappa}] = [\chi] = q^2/h = q/\Phi_0 = (q/t)/(\Phi_0/t) = \mathit{current}/\mathit{voltage} = 1/\mathit{resistance} \overset{\text{SI}}{=} 1/\Omega = \text{S}$ (for Siemens), but, moreover, they are measurable, too.

Two invariants of κ, a linear and a quadratic one, play a leading role: The twisted scalar

$$\alpha := \frac{1}{12}\,{\kappa_{ij}}^{ij} = \frac{1}{4!}\,\hat{\epsilon}_{ijkl}\,\chi^{[ijkl]} \tag{D.1.11}$$

invariant1

and the true scalar

$$\begin{aligned}\lambda^2 &:= -\frac{1}{4!}\,\kappa_{ij}{}^{kl}\kappa_{kl}{}^{ij} \\ &= -\frac{1}{96}\,\hat{\epsilon}_{ijkl}\,\hat{\epsilon}_{mnpq}\,\chi^{ijmn}\,\chi^{pqkl}\,.\end{aligned} \tag{D.1.12}$$

invariant2

In later applications we will see that it always fulfills $\lambda^2 > 0$. Note that $[\alpha] = [\lambda] = 1/$*resistance*. It is as if spacetime carried an intrinsic resistance or, the inverse of it, an intrinsic impedance (commonly called "wave resistance of the vacuum" or "vacuum impedance").

One could also build up invariants of order p according to the pattern $\kappa_{I_1}{}^{I_2}\,\kappa_{I_2}{}^{I_3}\ldots\kappa_{I_p}{}^{I_1}$, with $p = 1, 2, 3, 4\ldots$, but there seems no need to do so.

D.1.2 Extracting the Abelian axion

Right now (still without a metric), we can split off the totally antisymmetric part of χ^{ijkl} according to

$$\chi^{ijkl} = \widetilde{\chi}^{ijkl} + \alpha\,\epsilon^{ijkl}\,, \qquad \text{with} \qquad \widetilde{\chi}^{[ijkl]} = 0\,. \tag{D.1.13}$$

split

The invariant (D.1.11) shows up in the second term as a pseudo-scalar function $\alpha = \alpha(\sigma, x)$. Thus the *linearity ansatz* (D.1.6) eventually reads

$$H_{ij} = \frac{1}{4}\hat{\epsilon}_{ijmn}\,\widetilde{\chi}^{mnkl}\,F_{kl} + \alpha\,F_{ij}\,, \tag{D.1.14}$$

linear

with

$$\widetilde{\chi}^{\,mnkl} = -\widetilde{\chi}^{\,nmkl} = -\widetilde{\chi}^{\,mnlk} \quad \text{and} \quad \widetilde{\chi}^{\,[mnkl]} = 0\,. \tag{D.1.15}$$

chisymm

Besides the *Abelian axion* field α, we have 35 independent functions.

It is remarkable that the pseudo-scalar axion field α enters here as a quantity that does not interfere at all with the first four axioms of electrodynamics. Already at the pre-metric level, such a field emerges as a not unnatural companion of the electromagnetic field. Hence a possible axion field has a high degree of universality — after all, it arises, in the framework of our axiomatic approach, even before the metric field (Einstein's gravitational potential) comes into being.

Pseudo-scalars are also called axial scalars.[2] So far, our axial scalar $\alpha(x)$ is some kind of universal permittivity/permeability field. If one adds a kinetic term of the α-field to the purely electromagnetic Lagrangian, then $\alpha(x)$ becomes propagating and one can call it legitimately an Abelian[3] axion. The corresponding hypothetical particle[4] has spin $= 0$ and parity $= -1$.

The split (D.1.13) effectively introduces the linear operator $\widetilde{\boldsymbol{\kappa}} : \Lambda^2 X \longrightarrow \Lambda^2 X$ which acts in the space of two-forms similarly to $\boldsymbol{\kappa}$. Patterned after (D.1.4), its action on the $\mathcal{B}$'s reads

$$\widetilde{\boldsymbol{\kappa}}\left(dx^i \wedge dx^j\right) = \frac{1}{2}\widetilde{\kappa}_{kl}{}^{ij}\, dx^k \wedge dx^l \quad \text{or} \quad \widetilde{\boldsymbol{\kappa}}(\mathcal{B}^I) = \widetilde{\boldsymbol{\kappa}}_K{}^I \mathcal{B}^K. \tag{D.1.16}$$ sharp1

Here the linear operator matrix is evidently defined by

$$\widetilde{\kappa}_{kl}{}^{ij} := \frac{1}{2}\hat{\epsilon}_{klmn}\, \widetilde{\chi}^{mnij} \quad \text{or} \quad \widetilde{\kappa}_I{}^K := \hat{\epsilon}_{IM}\, \widetilde{\chi}^{MK}. \tag{D.1.17}$$ sharp2

Formulated in the 6D space of 2-forms, (D.1.14) then reads

$$H_I = \kappa_I{}^K F_K = \hat{\epsilon}_{IM}\, \widetilde{\chi}^{MK}\, F_K + \alpha\, F_I, \tag{D.1.18}$$ chiHF1

where

$$\hat{\epsilon}_{MK}\, \widetilde{\chi}^{MK} = \widetilde{\kappa}_K{}^K = 0\,. \tag{D.1.19}$$ also

---

[2] For a discussion of a possible primordial cosmological helicity, see G.B. Field, S.M. Carroll [4]; also magnetic helicity, that we addressed earlier in (B.3.17), is mentioned therein.

[3] In contrast to the axions related to *non*-Abelian gauge theories, see Peccei and Quinn [22], Weinberg [39], Wilczek and Moody [40, 17] and the reviews in Kolb and Turner [14] and Sikivie [31].

[4] Ni [18, 19] was the first to introduce such an axion field α in the context of the coupling of electromagnetism to gravity, see also deSabbata & Sivaram [29] and the references given there.

Summing up, the linear spacetime relation (D.1.6), see also (D.1.14), can be written as

$$\boxed{H = (\widetilde{\boldsymbol{\kappa}} + \alpha)\, F\,, \qquad \operatorname{Tr} \widetilde{\boldsymbol{\kappa}} = 0\,.} \tag{D.1.20}$$

lin1

Hence the Maxwell equations in this shorthand notation read

$$d\big[(\widetilde{\boldsymbol{\kappa}} + \alpha)F\big] = J\,, \qquad dF = 0\,. \tag{D.1.21}$$

Maxsharp

We can also execute the differentiation in the inhomogeneous equation and substitute the homogeneous one. Then we find for the inhomogeneous Maxwell equation

$$d\,\widetilde{\boldsymbol{\kappa}}(F) + d\alpha \wedge F = J\,. \tag{D.1.22}$$

Maxsharp'

As yet, the Abelian axion has not been found experimentally. In particular, it couldn't be traced in ring laser experiments.[5]

D.1.3 Fresnel equation

As soon as the constitutive law is specified, electrodynamics becomes a predictive theory and one can study various of its physical effects, such as the propagation of electromagnetic disturbances and, in particular, the phenomenon of wave propagation in vacuum.

In the theory of partial differential equations, the propagation of disturbances is described by the Hadamard discontinuities of solutions across a characteristic hypersurface S, the wave front.[6] One can locally define S by the equation $\Phi(x^i) = const.$ The Hadamard discontinuity of any function $\mathcal{F}(x)$ across the hypersurface S is determined as the difference $[\mathcal{F}]\,(x) := \mathcal{F}(x_+) - \mathcal{F}(x_-)$, where $x_\pm := \lim_{\varepsilon\to 0}(x \pm \varepsilon)$ are points on the opposite sides of $S \ni x$. We call $[\mathcal{F}]\,(x)$ the *jump* of the function $\mathcal{F}$ at x.

[5] See Cooper & Stedman [3] and Stedman [33] for a systematic and extended series of experiments.

[6] The corresponding theory was developped in detail by Hadamard [7] and Lichnerowicz [16], e.g..

An ordinary electromagnetic wave is a solution of Maxwell's vacuum equations $dH = 0$ and $dF = 0$ for which the derivatives of H and F are discontinuous across the wave front hypersurface S. In terms of H and F, we have on the characteristic hypersurface S

$$[H] = 0\,, \quad [dH] = q \wedge h\,, \tag{D.1.23}$$ this

$$[F] = 0\,, \quad [dF] = q \wedge f\,. \tag{D.1.24}$$ that

Here the 2-forms h, f describe the jumps (discontinuities) of the differentials of the electromagnetic field across S, and the wave-covector normal to the front is given by

$$q := d\Phi\,. \tag{D.1.25}$$

Equations (D.1.23) and (D.1.24) represent the Hadamard geometrical compatibility conditions. If we use Maxwell's vacuum equations $dH = 0$ and $dF = 0$, then (D.1.23) and (D.1.24) yield

$$q \wedge h = 0\,, \qquad q \wedge f = 0\,. \tag{D.1.26}$$ these

We can say something more about the jump h, if we apply the spacetime relation (D.1.20). Provided the components of the linear operator $\boldsymbol{\kappa}$ (that is, of $\widetilde{\boldsymbol{\kappa}}$ and α) are continuous across S, we find by differentiating (D.1.20) and using (D.1.23) and (D.1.24),

$$[dH] = [d\,\widetilde{\boldsymbol{\kappa}}(F)] + \alpha\,[dF] \; = q \wedge (\widetilde{\boldsymbol{\kappa}}(f) + \alpha\, f) = q \wedge h\,. \tag{D.1.27}$$

Accordingly, the jump equations (D.1.26) can be put into the form

$$q \wedge h = q \wedge \widetilde{\boldsymbol{\kappa}}(f) = 0\,, \qquad q \wedge f = 0\,. \tag{D.1.28}$$ those

Note that the axion drops out completely, i.e., $\widetilde{\boldsymbol{\kappa}}$ carries 35 independent components. If we multiply the two equations of (D.1.28) by q, both vanish identically. Hence only 3+3 equations turn out to be independent.

Spacetime components

We can rewrite these equations in spacetime components:

$$\begin{aligned} q \wedge \widetilde{\boldsymbol{\kappa}}(f) &= q_m \, dx^m \wedge \frac{1}{2} f_{ij} \, \widetilde{\boldsymbol{\kappa}}(dx^i \wedge dx^j) \\ &= \frac{1}{4} \, q_{[m} \, \widetilde{\kappa}_{kl]}{}^{ij} f_{ij} \, dx^m \wedge dx^k \wedge dx^l = 0, \end{aligned} \tag{D.1.29}$$

and

$$q \wedge f = \frac{1}{2} \, q_{[i} \, f_{jk]} \, dx^i \wedge dx^j \wedge dx^k = 0 \tag{D.1.30}$$

As a consequence, we find

$$\epsilon^{ijkl} \, q_j \, h_{kl} = \widetilde{\chi}^{ijkl} \, q_j \, f_{kl} = 0 \, , \qquad \epsilon^{ijkl} \, q_j \, f_{kl} = 0 \, . \tag{D.1.31}$$ 4Dwave

If the constitutive tensor density $\widetilde{\chi}^{ijkl}$ is prescribed, the set in (D.1.31) represents homogeneous algebraic equations for the 2-forms f_{ij}.

Components in the space of 2-forms

Alternatively, we can decompose $\widetilde{\boldsymbol{\kappa}}(f)$ and f in (D.1.28) with respect to the $\mathcal{B}^I$-basis of the M^6. Then (D.1.28) can be rewritten as

$$q \wedge h_I \mathcal{B}^I = q \wedge \widetilde{\kappa}_I{}^K f_K \mathcal{B}^I = 0 \, , \qquad q \wedge f_I \mathcal{B}^I = 0 \, . \tag{D.1.32}$$ those1

Before we start to exploit these equations, it will be convenient to make in this picture the presence of electric and magnetic pieces more pronounced. For this reason, we need to recall, on the one hand, the (1+3)-decompositions (B.4.5) and (B.4.6) of the excitations and the field strengths, and to compare these, on the other hand, with the the decompositions of H and F with respect to the $\mathcal{B}^I$ basis:

$$H \;=\; H_I \, \mathcal{B}^I = -\mathcal{H} \wedge d\sigma + \mathcal{D} \, , \tag{D.1.33}$$

$$F \;=\; F_I \, \mathcal{B}^I = \;\; E \wedge d\sigma + B \, . \tag{D.1.34}$$

Since every longitudinal (spatial) 1-form can be decomposed with respect to the natural coframe dx^a, whereas every 2-form can be conveniently expanded with respect to the ϵ-dual 2-form basis $\hat{\epsilon}_a$, we have $(a, b, \dots = 1, 2, 3)$

$$\mathcal{H} = \mathcal{H}_a \, dx^a \,, \quad E = E_a \, dx^a \,, \tag{D.1.35}$$

and

$$\mathcal{D} = \mathcal{D}^c \, \hat{\epsilon}_c \,, \quad B = B^c \, \hat{\epsilon}_c \,. \tag{D.1.36}$$

We substitute this into (D.1.33), (D.1.34) and find

$$H = H_I \, B^I = \quad \mathcal{H}_a \, \beta^a + \mathcal{D}^b \, \hat{\epsilon}_b, \tag{D.1.37}$$
comp1

$$F = F_I \, B^I = - E_a \, \beta^a + B^b \, \hat{\epsilon}_b. \tag{D.1.38}$$
comp2

Then the spacetime relation $H = \widetilde{\boldsymbol{\kappa}}(F)$, in components $H_I = \widetilde{\kappa}_I{}^K F_K$, reads

$$\begin{pmatrix} \mathcal{H}_a \\ \mathcal{D}^a \end{pmatrix} = \begin{pmatrix} \widetilde{C}^b{}_a & \widetilde{B}_{ab} \\ \widetilde{A}^{ab} & \widetilde{D}_b{}^a \end{pmatrix} \begin{pmatrix} -E_b \\ B^b \end{pmatrix} . \tag{D.1.39}$$
CR

For the jumps h and f, we find analogous equations. Instead of (D.1.37) and (D.1.38), we have

$$h = \quad h_a \, \beta^a + d^a \, \hat{\epsilon}_a \,, \tag{D.1.40}$$
dish

$$f = -e_a \, \beta^a + b^a \, \hat{\epsilon}_a \,, \tag{D.1.41}$$
disf

and the equation derived from (D.1.39) is

$$\begin{pmatrix} h_a \\ d^a \end{pmatrix} = \begin{pmatrix} \widetilde{C}^b{}_a & \widetilde{B}_{ab} \\ \widetilde{A}^{ab} & \widetilde{D}_b{}^a \end{pmatrix} \begin{pmatrix} -e_b \\ b^b \end{pmatrix} . \tag{D.1.42}$$
CR1

Now we turn to (D.1.32). We insert the wave covector in its expanded form

$$q = q_0 \, d\sigma + q_a dx^a \tag{D.1.43}$$
wavesplit

and use (D.1.40) and (D.1.41):

$$q_0\, d^a - \epsilon^{abc}\, q_b\, h_c = 0, \quad q_0\, b^a + \epsilon^{abc}\, q_b\, e_c = 0, \tag{D.1.44}$$ geo2

$$q_a\, d^a = 0, \quad q_a\, b^a = 0. \tag{D.1.45}$$ geo3

In this system, which is a component version of (D.1.26), only the 6 equations (D.1.44) are independent. This has already been foreseen above. Assuming that $q_0 \neq 0$, one finds that the equations (D.1.45) are trivially satisfied if one substitutes (D.1.44) into them. Note that the characteristics with $q_0 = 0$ do not have an intrinsic meaning for the evolution equations, since they obviously depend on the choice of the coordinates.

We can now substitute d^a and h_a from (D.1.42) into $(\text{D.1.44})_1$:

$$q_0 \left(-\widetilde{A}^{ab}\, e_b + \widetilde{D}_b{}^a\, b^b\right) = \epsilon^{abc} q_b \left(-\widetilde{C}^d{}_c\, e_d + \widetilde{B}_{cd}\, b^d\right) \tag{D.1.46}$$ 3Dwave

Then we eliminate b^b by means of $(\text{D.1.44})_2$:

$$\left[q_0^2\, \widetilde{A}^{ab} - q_0 q_c \left(\epsilon^{acd}\, \widetilde{C}^b{}_d + \epsilon^{bcd}\, \widetilde{D}_d{}^a\right) + q_c q_d\, \epsilon^{ace} \epsilon^{bdf}\, \widetilde{B}_{ef}\right] e_b = 0. \tag{D.1.47}$$ algeq

This homogeneous linear algebraic equation for e_b has a nontrivial solution provided the determinant $\mathcal{W} := \det W$ of its coefficient matrix

$$W^{ab} := q_0^2\, \widetilde{A}^{ab} + q_0\, Y^{ab} + Z^{ab} \tag{D.1.48}$$ Fresnelmat

vanishes, where

$$Y^{ab} := \left(\epsilon^{acd}\, \widetilde{C}^b{}_c + \epsilon^{bcd}\, \widetilde{D}_c{}^a\right) q_d, \tag{D.1.49}$$ Ydef

$$Z^{ab} := \epsilon^{ace} \epsilon^{bdf}\, \widetilde{B}_{cd}\, q_e q_f. \tag{D.1.50}$$ Zdef

We have then, with the definition of the determinant for a 3×3 matrix,

$$\mathcal{W} = \det W = \frac{1}{6}\, \hat{\epsilon}_{abc}\, \hat{\epsilon}_{def}\, W^{ad} W^{be} W^{cf} = 0\,. \tag{D.1.51}$$ deteq0

This is the Fresnel equation that is of central importance in any wave propagation analysis. It determines the geometry of the wave normals in terms of the 35 independent constitutive components of the 3×3 matrices $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$.

D.1.4 Analysis of the Fresnel equation

The following properties are evident from the definitions above:

$$Z^{ab} q_b = 0, \qquad Z^{ab} q_a = 0, \qquad Y^{ab} q_a q_b = 0. \tag{D.1.52}$$ YZprop

Immediate inspection shows that the determinant reads:

$$\begin{aligned}
\mathcal{W} = {} & q_0^6 \, \det \widetilde{A} + q_0^5 \, \frac{1}{2} \, \hat{\epsilon}_{abc} \, \hat{\epsilon}_{def} \widetilde{A}^{ad} \widetilde{A}^{be} Y^{cf} \\
& + q_0^4 \, \frac{1}{2} \, \hat{\epsilon}_{abc} \, \hat{\epsilon}_{def} \left(Z^{ae} \widetilde{A}^{be} \widetilde{A}^{cf} + Y^{ad} Y^{be} \widetilde{A}^{cf} \right) \\
& + q_0^3 \left(\det Y + \hat{\epsilon}_{abc} \, \hat{\epsilon}_{def} Z^{ad} \widetilde{A}^{be} Y^{cf} \right) \\
& + q_0^2 \, \frac{1}{2} \, \hat{\epsilon}_{abc} \, \hat{\epsilon}_{def} \left(Z^{ad} Y^{be} Y^{cf} + Z^{ad} Z^{be} \widetilde{A}^{cf} \right) \\
& + q_0 \, \frac{1}{2} \hat{\epsilon}_{abc} \, \hat{\epsilon}_{def} Z^{ad} Z^{be} Y^{cf} + \det Z.
\end{aligned} \tag{D.1.53}$$ determinant

We have $\det Z = 0$ since the matrix Z has null eigenvectors. Hence, the last term drops out completely. Furthermore, by means of (D.1.50), we find

$$\hat{\epsilon}_{abc} \, \hat{\epsilon}_{def} Z^{ad} = \widetilde{B}_{be} q_c q_f - \widetilde{B}_{bf} q_c q_e - \widetilde{B}_{ce} q_b q_f + \widetilde{B}_{cf} q_b q_e . \tag{D.1.54}$$ epsepsZ

If we multiply this equation by $Z^{be} Y^{cf}$, we recover the term linear in q_0 of the last line in (D.1.53). Then, by using (D.1.52), we straightforwardly recognize that it vanishes identically.

Consequently, we have verified that the determinant $\mathcal{W}$ factorizes into a product of q_0^2 and a 4th order polynomial:

$$\begin{aligned}
\mathcal{W} = q_0^2 \, \big(& q_0^4 M + q_0^3 q_a \, M^a + q_0^2 q_a q_b \, M^{ab} \\
& + q_0 q_a q_b q_c \, M^{abc} + q_a q_b q_c q_d \, M^{abcd} \big) = 0 \, .
\end{aligned} \tag{D.1.55}$$ detlin

Thus, the Fresnel equation describes ultimately a 4th order surface[7] and not one of 6th order, as it appears from (D.1.53).

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[7] Numerical evaluations and corresponding plots of Fresnel wave surfaces for a constitutive tensor including optical activity and Faraday rotation have been presented by Kiehn et al.[13].

By using (D.1.54), (D.1.49), (D.1.50) and (D.1.52), we find for the different pieces in (D.1.53):

$$\begin{aligned}\frac{1}{2}\,\hat\epsilon_{abc}\,\hat\epsilon_{def}\,Z^{ad}Y^{be}Y^{cf} &= -\,\widetilde{B}_{bf}q_cq_eY^{be}Y^{cf}\\ &= -\,q_aq_bq_cq_d\,\epsilon^{efa}\epsilon^{ghb}\widetilde{D}_e{}^c\widetilde{C}^d{}_gB_{hf}, \qquad \text{(D.1.56)}\\ \frac{1}{2}\,\hat\epsilon_{abc}\,\hat\epsilon_{def}\,Z^{ad}Z^{be}\widetilde{A}^{cf} &= \frac{1}{2}\widetilde{B}_{be}q_cq_fZ^{be}\widetilde{A}^{cf}\\ &= \frac{1}{2}q_aq_bq_cq_d\,\epsilon^{efa}\epsilon^{ghb}\widetilde{B}_{eg}B_{fh}\widetilde{A}^{cd}. \qquad \text{(D.1.57)}\end{aligned}$$

Furthermore,

$$\begin{aligned}\hat\epsilon_{abc}\,\hat\epsilon_{def}\,Z^{ad}\widetilde{A}^{be}Y^{cf} = q_aq_bq_c\,\epsilon^{dea}\Big[\widetilde{A}^{bc}\left(\widetilde{B}_{fd}\widetilde{D}_e{}^f + \widetilde{B}_{df}\widetilde{C}^f{}_e\right)\\ -\,\widetilde{A}^{cf}\widetilde{C}^b{}_e\widetilde{B}_{df} - \widetilde{A}^{fc}\widetilde{D}_e{}^b\widetilde{B}_{fd}\Big]. \qquad \text{(D.1.58)}\end{aligned}$$

Eventually, a somewhat lengthy calculation yields:

$$\begin{aligned}\det Y &= \frac{1}{6}\,\hat\epsilon_{abc}\,\hat\epsilon_{def}\,Y^{ad}Y^{be}Y^{cf}\\ &= q_aq_bq_c\,\epsilon^{dea}\left(\widetilde{D}_f{}^b\widetilde{C}^c{}_e\widetilde{C}^f{}_d + \widetilde{C}^b{}_f\widetilde{D}_e{}^c\widetilde{D}_d{}^f\right). \qquad \text{(D.1.59)}\end{aligned}$$

As a result, we can finalize the computation of the coefficients of the Fresnel equation. We find explicitly,

$$M := \det \widetilde{A}\,, \tag{D.1.60}$$
ma0

$$M^{a} := -\hat{\epsilon}_{bcd}\left(\widetilde{A}^{ab}\widetilde{A}^{ec}\widetilde{C}^{d}{}_{e} + \widetilde{A}^{ba}\widetilde{A}^{ce}\widetilde{D}_{e}{}^{d}\right)\,, \tag{D.1.61}$$
ma1

$$\begin{aligned} M^{ab} := {} & \frac{1}{2}\,\widetilde{A}^{(ab)}\left[(\widetilde{C}^{d}{}_{d})^2 + (\widetilde{D}_{c}{}^{c})^2 - (\widetilde{C}^{c}{}_{d} + \widetilde{D}_{d}{}^{c})(\widetilde{C}^{d}{}_{c} + \widetilde{D}_{c}{}^{d})\right] \\ & + (\widetilde{C}^{d}{}_{c} + \widetilde{D}_{c}{}^{d})(\widetilde{C}^{(a}{}_{d}\widetilde{A}^{b)c} + \widetilde{A}^{c(a}\widetilde{D}_{d}{}^{b)}) \\ & - \widetilde{C}^{d}{}_{d}\widetilde{C}^{(a}{}_{c}\widetilde{A}^{b)c} - \widetilde{A}^{c(a}\widetilde{D}_{c}{}^{b)}\widetilde{D}_{d}{}^{d} - \widetilde{A}^{cd}\widetilde{C}^{(a}{}_{c}\widetilde{D}_{d}{}^{b)} \\ & + \left(\widetilde{A}^{(ab)}\widetilde{A}^{cd} - \widetilde{A}^{c(a}\widetilde{A}^{b)d}\right)\widetilde{B}_{cd}\,, \end{aligned} \tag{D.1.62}$$
ma2

$$\begin{aligned} M^{abc} := {} & \epsilon^{de(c}\Big[\widetilde{B}_{fd}\left(\widetilde{A}^{ab)}\,\widetilde{D}_{e}{}^{f} - \widetilde{A}^{|f|a}\,\widetilde{D}_{e}{}^{b)}\right) \\ & + \widetilde{B}_{df}\left(\widetilde{A}^{ab)}\,\widetilde{C}^{f}{}_{e} - \widetilde{C}^{a}{}_{e}\widetilde{A}^{b)f}\right) \\ & + \widetilde{C}^{a}{}_{f}\,\widetilde{D}_{e}{}^{b)}\,\widetilde{D}_{d}{}^{f} + \widetilde{D}_{f}{}^{a}\,\widetilde{C}^{b)}{}_{e}\,\widetilde{C}^{f}{}_{d}\Big]\,, \end{aligned} \tag{D.1.63}$$
ma3

$$M^{abcd} := \epsilon^{(a|ef}\epsilon^{gh|b}\,\widetilde{B}_{eg}\left(\tfrac{1}{2}\,\widetilde{A}^{cd)}\widetilde{B}_{fh} - \widetilde{C}^{c}{}_{f}\widetilde{D}_{h}{}^{d)}\right)\,. \tag{D.1.64}$$
ma4

The M's are all cubic in the components of $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$. We defined the M's as completely symmetric expressions, that is, $M^{a_1\dots a_p} = M^{(a_1\dots a_p)}$, $p = 2, 3, 4$, since only these pieces contribute to the Fresnel equation. Since a totally symmetric tensor of rank p has $\binom{n+p-1}{p} = \binom{n-1+p}{n-1}$ independent components, the M's altogether carry $1 + 3 + 6 + 10 + 15 = 35$ independent components.

Since $q_0 \neq 0$, one can delete the first factor in (D.1.55). Thus we find finally that the wave covector q_i lies, in general, on a *4th order* (or quartic) *surface*. Of course, a quartic wave surface for light propagation is unphysical, at least at the present epoch of the universe. It is different from the 2nd order structure of the light*cone*, which arises from the quartic surface only under particular circumstances. Further below we will demonstrate which type of constraint can enforces such a reduction.[8] Nevertheless, if one wanted to generalize the lightcone structure for the very early universe, for example, then there would be no need to turn to nonlinear electrodynamics (see Chapter E.2); also a linear spacetime relation can support nontrivial generalizations of the conventional (2nd order) lightcone to a quartic wave surface.

Consider the expression in the parenthesis of the Fresnel equation (D.1.55). If we recall that the wave covector splits according to (D.1.43), then this expression can be rewritten in a 4-dimensional invariant form,

$$\boxed{\mathcal{G}^{ijkl} q_i q_j q_k q_l = 0, \qquad i, j, \dots = 0, 1, 2, 3\,.} \tag{D.1.65}$$

Fresnel

The totally symmetric fourth order tensor density $\mathcal{G}^{ijkl}$ of weight $+1$ has 35 independent components, exactly the same number as those of the M's. We can express $\mathcal{G}^{ijkl}$ componentwise in terms of the M's. Start with 4 q_0's. Then we find successively, by comparing (D.1.55) with (D.1.65),

$$\mathcal{G}^{0000} = M\,, \quad \mathcal{G}^{000a} = \frac{1}{4} M^a\,, \quad \mathcal{G}^{00ab} = \frac{1}{6} M^{ab}\,,$$
$$\mathcal{G}^{0abc} = \frac{1}{4} M^{abc}\,, \quad \mathcal{G}^{abcd} = M^{abcd}\,. \tag{D.1.66}$$

compare

In the end, we have expressed the 35 independent components of $\mathcal{G}^{ijkl}$ in terms of the 35 independent components of the M's; and the M's can be expressend in terms of the 35 independent components of $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$. These constitute the tensor $\widetilde{\kappa}_{ij}{}^{kl}$, see (D.1.39), which, according to (D.1.17), is equivalent to $\widetilde{\chi}^{ijkl}$. The conclusion from this backtracing process is that $\mathcal{G}^{ijkl}$ should be expressible in terms of $\widetilde{\chi}^{ijkl}$. The cubic Tamm-Rubilar formula is the answer:

$$\boxed{\mathcal{G}^{ijkl} := \frac{1}{4!}\,\hat{\epsilon}_{mnpq}\,\hat{\epsilon}_{rstu}\,\widetilde{\chi}^{mnr(i}\,\widetilde{\chi}^{j|ps|k}\,\widetilde{\chi}^{l)qtu}\,.} \tag{D.1.67}$$

G4

[8] Earlier, the relation between the fourth- and the second-order wave geometry was studied by Tamm [34] for a special case of a linear constitutive law. He also introduced a 'fourth-order metric' of the type of our $\mathcal{G}$ tensor density, see (D.1.67).

Here the total symmetrization is extended only over the four indices i, j, k, l with all the dummy (or dead) summation indices excluded. Although eq. (D.1.67) can be verified by means of computer algebra, its covariant analytic derivation remains an interesting and difficult problem. Our readers are invited to solve it.

By definition, $\mathcal{G}^{ijkl}(\widetilde{\chi})$ does not depend on the axion. We saw that in our analysis à la Hadamard the axion drops out, see the remark after (D.1.28). It is consistent with that result that $\mathcal{G}^{ijkl}(\chi) = \mathcal{G}^{ijkl}(\widetilde{\chi} + \alpha\,\epsilon) = \mathcal{G}^{ijkl}(\widetilde{\chi})$, as can be seen by direct calculation.

D.2

Electric-magnetic reciprocity switched on

The linear spacetime relation of the last chapter is required to obey electric-magnetic reciprocity. This implies an almost complex structure on spacetime thereby reducing the constitutive tensor $\widetilde{\boldsymbol{\kappa}}$ from 35 to 18 independent components.

D.2.1 Reciprocity implies closure

The linear spacetime relation leaves us still with 35+1 independent components of the tensor $\boldsymbol{\kappa}$. Clearly we need a new method to constrain $\boldsymbol{\kappa}$ in some way. An obvious choice is to require electric-magnetic reciprocity for (D.1.20). We have discovered electric-magnetic reciprocity as a property of the energy-momentum current ${}^{\mathrm{k}}\Sigma_\alpha$ of the electromagnetic field. Why should't we apply it to (D.1.20), too?

The electric-magnetic reciprocity transformation (B.5.15)

$$H \to \zeta F\,, \qquad F \to -\frac{1}{\zeta}\,H\,, \tag{D.2.1}$$ duality2

can alternatively be written as

$$\begin{pmatrix} H \\ F \end{pmatrix} \to \begin{pmatrix} 0 & \zeta \\ -\frac{1}{\zeta} & 0 \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix} = \begin{pmatrix} \zeta F \\ -\frac{1}{\zeta} H \end{pmatrix} . \qquad \text{(D.2.2)}$$ duality2a

If

$$W := \begin{pmatrix} 0 & \zeta \\ -\frac{1}{\zeta} & 0 \end{pmatrix} , \quad \text{then} \quad W^{-1} = \begin{pmatrix} 0 & -\zeta \\ \frac{1}{\zeta} & 0 \end{pmatrix} . \qquad \text{(D.2.3)}$$ duality2b

Let us perform an electric-magnetic reciprocity transformation in (D.1.20). By definition, the reciprocity transformation *commutes* with the linear operator $\widetilde{\boldsymbol{\kappa}}$. Then we find

$$\zeta F = (\widetilde{\boldsymbol{\kappa}} + \alpha) \left(-\frac{H}{\zeta} \right) \quad \text{or} \quad \widetilde{\boldsymbol{\kappa}}(H) = -\zeta^2 \, F - \alpha \, H \, . \qquad \text{(D.2.4)}$$ duality3

On the other hand, we can also apply the operator $\widetilde{\boldsymbol{\kappa}}$ to (D.1.20). Because $\widetilde{\boldsymbol{\kappa}}$ commutes with 0-forms, we get

$$\widetilde{\boldsymbol{\kappa}}(H) = (\widetilde{\boldsymbol{\kappa}} \, \widetilde{\boldsymbol{\kappa}} + \alpha \, \widetilde{\boldsymbol{\kappa}}) \, F \, . \qquad \text{(D.2.5)}$$ lin3

If we postulate electric-magnetic reciprocity of the linear law (D.1.20), then, as a comparison of (D.2.5) and (D.2.4) shows, we have to assume additionally

$$\widetilde{\boldsymbol{\kappa}} \, \widetilde{\boldsymbol{\kappa}} = -\zeta^2 \, \mathbf{1}_6, \qquad \alpha = 0 \, . \qquad \text{(D.2.6)}$$ close1

We call $\widetilde{\boldsymbol{\kappa}} \, \widetilde{\boldsymbol{\kappa}} = -\zeta^2 \, \mathbf{1}_6$ the *closure relation*[1] since applying the operator $\widetilde{\boldsymbol{\kappa}}$ twice, we come back, up to a negative function, to the identity $\mathbf{1}_6$ ($= \delta^{ij}_{kl}$, in components). In this sense, the operation closes.

At the same time, (D.2.6) tells us that the spacetime relation is *not* electric-magnetic reciprocal for an arbitrary transformation function ζ (like the energy-momentum current is). The linear operator $\widetilde{\boldsymbol{\kappa}}$ is based on the measurable components $\widetilde{\kappa}_{ij}{}^{kl}$. If

[1] Toupin [35], for $\zeta^2 = \text{const}$, just called the closure relation "electric and magnetic reciprocity".

applied twice, as in (D.2.6), there must not emerge an arbitrary function. In other words, we can solve $(\mathrm{D}.2.6)_1$ for ζ by taking its trace,

$$\zeta^2 = -\frac{1}{6}\,\mathrm{Tr}\,(\widetilde{\boldsymbol{\kappa}}\,\widetilde{\boldsymbol{\kappa}}) = -\frac{1}{24}\,\widetilde{\kappa}_{kl}{}^{ij}\,\widetilde{\kappa}_{ij}{}^{kl} = \lambda^2\,, \tag{D.2.7}$$ zetasquare

with the quadratic invariant λ^2 of (D.1.12).

D.2.2 Almost complex structure

Now we can factorize the "constitutive" matrix (D.1.13) by means of the dimensionfull function λ according to

$$\widetilde{\chi}^{ijkl} =: \lambda\,\overset{\circ}{\chi}{}^{ijkl}\,. \tag{D.2.8}$$ circle

Here $[\lambda] = 1/\mathit{resistance}$, as we already know, and $\overset{\circ}{\chi}{}^{ijkl}$ is a *dimensionless* tensor with the same symmetries as displayed in (D.1.15). This effectively defines a new operator $\mathbf{J}$ via

$$\boxed{\widetilde{\boldsymbol{\kappa}} =: \lambda\,\mathbf{J}\,.} \tag{D.2.9}$$ close1a'

For $\mathbf{J}$ the closure reads

$$\boxed{\mathbf{J}\mathbf{J} = -\,\mathbf{1}_6.} \tag{D.2.10}$$ close1a

As we will see, the minus sign is very decisive: It will eventually yield the Lorentzian signature of the metric of spacetime. With the closure relation (D.2.10), the operator $\mathbf{J}$ is an *almost complex structure* on the space M^6 of 2-forms, as discussed in Sec. A.1.11.

If we apply $\mathbf{J}$ to the 2-form basis, see (D.1.16), we find

$$\mathbf{J}(\mathit{B}^I) = \frac{1}{\lambda}\,\widetilde{\boldsymbol{\kappa}}(\mathit{B}^I) = \frac{1}{\lambda}\,\widetilde{\kappa}_K{}^I(\mathit{B}^K) = \mathrm{J}_K{}^I\mathit{B}^K\,. \tag{D.2.11}$$ Jdecomp

A comparison with (D.1.39) allows to express $\mathbf{J}$ in terms of the constitutive functions according to

$$\left(\mathrm{J}_I{}^K\right) = \frac{1}{\lambda}\begin{pmatrix} \widetilde{C}^a{}_b & \widetilde{A}^{ab} \\ \widetilde{B}_{ab} & \widetilde{D}_a{}^b \end{pmatrix} =: \begin{pmatrix} C^a{}_b & A^{ab} \\ B_{ab} & D_a{}^b \end{pmatrix}. \tag{D.2.12}$$ Jmatrix

Because of (D.2.10), the 3×3 blocks A, B, C, D are constrained by

$$AB + C^2 = -\mathbf{1}_3\,, \qquad \text{(D.2.13)}$$ almostclose1m

$$CA + AD = 0\,, \qquad \text{(D.2.14)}$$ almostclose2m

$$BC + DB = 0\,, \qquad \text{(D.2.15)}$$ almostclose3m

$$BA + D^2 = -\mathbf{1}_3\,. \qquad \text{(D.2.16)}$$ almostclose4m

D.2.3 Algebraic solution of the closure relation

We are able to solve this closure relation. Assume that $\det B \neq 0$. Consider (D.2.15). Then we can make an ansatz for the matrix C, namely

$$C = B^{-1}K. \qquad \text{(D.2.17)}$$ CDKK

We substitute this into (D.2.15) and find

$$D = -KB^{-1}. \qquad \text{(D.2.18)}$$ KK'

Next, we straightforwardly solve (D.2.13) with respect to A:

$$A = -B^{-1} - \left(B^{-1}K\right)^2 B^{-1}. \qquad \text{(D.2.19)}$$ ABK

Now we turn to (D.2.14). Eqs.(D.2.17) and (D.2.19) yield,

$$CA = -B^{-1}KB^{-1} - \left(B^{-1}K\right)^3 B^{-1}, \qquad \text{(D.2.20)}$$ CAAD

$$AD = B^{-1}KB^{-1} + \left(B^{-1}K\right)^3 B^{-1}. \qquad \text{(D.2.21)}$$

Thus, we conclude that (D.2.14) is satisfied. Finally, (D.2.14) is left for consideration. From (D.2.19) and $(\text{D.2.17})_2$, we obtain:

$$BA = -1 - \left(KB^{-1}\right)^2, \qquad \text{(D.2.22)}$$

$$D^2 = \left(KB^{-1}\right)^2. \qquad \text{(D.2.23)}$$

Thus, in view of (D.3.28), the equation (D.2.16) is also fulfilled.

Summing up, we have derived the general solution of the closure system (D.2.13)-(D.2.16). It reads,

$$A = -B^{-1} - \left(B^{-1}K\right)^2 B^{-1}, \tag{D.2.24}$$
summarycompA

$$C = B^{-1}K, \tag{D.2.25}$$
summarycompC

$$D = -KB^{-1}, \tag{D.2.26}$$
summarycompD

or, if we put this in 6×6 matrix form,

$$\mathbf{J} = \begin{pmatrix} C & A \\ B & D \end{pmatrix} = \begin{pmatrix} B^{-1}K & -\left[1 + (B^{-1}K)^2\right] B^{-1} \\ B & -KB^{-1} \end{pmatrix}. \tag{D.2.27}$$
summarymB1

By squaring $\mathbf{J}$, one can directly verify the closure relation (D.2.10). The arbitrary matrices B and K parametrize the solution which thus has $2 \times 9 = 18$ independent degrees of freedom.

Clearly, it would be possible to substitute K via $K = BC$, but the matrix K will turn out to be particularly useful in Chap. D.3. If we measure the elements of the 3×3 matrices $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ of $\widetilde{\boldsymbol{\kappa}}$ and thereby, according to (D.2.12), also those of A, B, C, D, then closure is only guaranteed provided the relations (D.2.24) to (D.2.26) are fulfilled. Therefore, closure has a well-defined operational meaning.

If we assume that $\det A \neq 0$, then an analogous derivation leads to

$$\mathbf{J} = \begin{pmatrix} C & A \\ B & D \end{pmatrix} = \begin{pmatrix} \widehat{K}A^{-1} & A \\ -A^{-1}[1 + (\widehat{K}A^{-1})^2] & -A^{-1}\widehat{K} \end{pmatrix}. \tag{D.2.28}$$
summarymA1

Before turning to the different subcases, a word to the physics of all of this appears to be in order. We know from experiments in vacuum that $\mathcal{D} \sim \varepsilon_0 \, E$ and $\mathcal{H} \sim \mu_0 \, B$. If we compare this with the spacetime relation (D.1.39), then we recognize that ε_0 is related the 3×3 matrix $\widetilde{A}$ and μ_0 to the 3×3 matrix $\widetilde{B}$. Therefore it is safe to assume that $\det A \neq 0$ and $\det B \neq 0$. From this practical point of view the other subcases don't seem to be of much interest.

We could substitute the matrices A, C, D of (D.2.24) to (D.2.26) into the M's (D.1.60) to (D.1.64) of the Fresnel equation and could discuss the corresponding consequences. However, we get a much more decisive restructuring of the Fresnel equation if we employ, in addition, the symmetry assumption which we will now turn to.

D.3

Symmetry switched on additionally

Having found an almost complex structure on spacetime by linearity and electric-magnetic reciprocity, how can we recover a spacetime metric eventually? By requiring symmetry of the constitutive tensor thereby reducing its independent components to 9. Thus, a lightcone is defined at each point of spacetime.

D.3.1 Lagrangian and symmetry

Besides linearity and electric-magnetic reciprocity of the electromagnetic spacetime relation, we require the operator $\boldsymbol{\kappa}$ to be *symmetric*,

$$\boldsymbol{\kappa}(\phi) \wedge \psi = \phi \wedge \boldsymbol{\kappa}(\psi)\,, \qquad \text{(D.3.1)}$$ symm

for arbitrary twisted or untwisted 2-forms ϕ and ψ. Similar relations are valid for $\widetilde{\boldsymbol{\kappa}}$ and $\mathbf{J}$.

This can be motivated in that we usually assume that a Lagrange 4-form exists for a fundamental theory, as we discussed in Sec. B.5.4. If we do this in the context of our linear spacetime relation, then, because of $H = -\,\partial V/\partial F$, the Lagrangian must be quadratic in F. Thus we find

$$V = -\frac{1}{2}\,H \wedge F = -\frac{1}{8}\,H_{ij}F_{pq}\,dx^i \wedge dx^j \wedge dx^p \wedge dx^q$$
$$= -\frac{1}{32}\,\hat{\epsilon}_{ijmn}\,\chi^{mnkl}\,F_{kl}F_{pq}\,dx^i \wedge dx^j \wedge dx^p \wedge dx^q\,. \quad \text{(D.3.2)}$$

We rewrite the exterior products with the Levi-Civita symbol:

$$V = -\frac{1}{8}\,\chi^{ijkl}\,F_{ij}F_{kl}\,dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\,. \qquad \text{(D.3.3)}$$

The components of the field strength F enter in a *symmetric* way. Therefore, without loss of generality, we can impose the symmetry condition

$$\chi^{ijkl} = \chi^{klij} \qquad \text{or} \qquad \chi^{IK} = \chi^{KI} \qquad \text{(D.3.4)}$$ sym

reducing them to 21 independent functions at this stage. This condition is equivalent to (D.3.1). Thus we split χ into $36 = 21 + 15$ independent components and require the vanishing of 15 of them.

The split (D.1.13) of the Abelian axion obviously does not disturb the symmetry property. Thus, $36 = 20 + 1 + 15$ and the tensor $\widetilde{\chi}^{\,ijkl}$ has the same algebraic symmetries and the same number of 20 independent components as a curvature tensor in a 4-dimensional Riemannian spacetime. Of course, the closure relation further restricts the remaining 20 components.

In the language of M^6 space, the 6×6 matrix $\widetilde{\chi}^{IK}$ is symmetric, and it can be represented with the help of 3×3 matrices $\widetilde{A}, \widetilde{B}, \widetilde{C}$ as

$$\boxed{\widetilde{\chi}^{IK} = \widetilde{\chi}^{KI}} = \begin{pmatrix} \widetilde{B} & \widetilde{C}^{\mathrm{T}} \\ \widetilde{C} & \widetilde{A} \end{pmatrix}, \quad \widetilde{A} = \widetilde{A}^{\mathrm{T}}, \quad \widetilde{B} = \widetilde{B}^{\mathrm{T}}\,. \quad \text{(D.3.5)}$$ xixi1

Accordingly, for the functions in (D.1.18), we find

$$\kappa_I{}^K = \hat{\epsilon}_{IM}\,\chi^{MK} = \begin{pmatrix} 0 & \mathbf{1}_3 \\ \mathbf{1}_3 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{B} & \alpha + \widetilde{C}^{\mathrm{T}} \\ \alpha + \widetilde{C} & \widetilde{A} \end{pmatrix}$$
$$= \begin{pmatrix} \alpha + \widetilde{C} & \widetilde{A} \\ \widetilde{B} & \alpha + \widetilde{C}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \widetilde{C} & \widetilde{A} \\ \widetilde{B} & \widetilde{C}^{\mathrm{T}} \end{pmatrix} + \alpha\,\mathbf{1}_6\,. \quad \text{(D.3.6)}$$ xi0

Note that we have $D = C^{\mathrm{T}}$ in this case.

The spacetime relation (D.1.20) with the symmetry (D.3.1)

$$H = (\widetilde{\boldsymbol{\kappa}} + \alpha)\, F\,, \quad \text{with} \quad \boldsymbol{\kappa}(F) \wedge F = F \wedge \boldsymbol{\kappa}(F)\,, \tag{D.3.7}$$

lin1a

leads to the Lagrangian

$$V = -\frac{1}{2}\left[F \wedge \widetilde{\boldsymbol{\kappa}}(F) + \alpha\, F \wedge F\right]\,. \tag{D.3.8}$$

axlagrangian

Hence, as a look at (D.1.12) will show, the coupling term to the axion reads

$$-\frac{1}{2}\,\alpha\, F \wedge F = -\alpha\, E \wedge B \wedge d\sigma\,, \tag{D.3.9}$$

with the electric field strenght 1-form E and the magnetic 2-form B in 3D. This term can also be rewritten as an exact form plus a supplementary term:

$$-\frac{1}{2}\,\alpha\, F \wedge F = d\left(-\frac{1}{2}\,\alpha F \wedge A\right) - \frac{1}{2}\, F \wedge A \wedge d\alpha\,. \tag{D.3.10}$$

axlagrangian1

For the special case of $\alpha = const$, we are left with a pure surface term.

D.3.2 Duality operator and metric

Accordingly, it is clear that the existence of a Lagrangian enforces the Onsager type of symmetry (D.3.1). We don't know whether the reverse is also true. What we do know from Sec. C.2.4, however, is that, in the framework of an almost complex structure of (D.2.9), the symmetry (D.3.1) introduces a duality operator. Therefore, we define a duality operator according to

$$\boxed{{}^{\#} := \mathbf{J} = \frac{1}{\lambda}\,\widetilde{\boldsymbol{\kappa}}\,, \quad \text{with} \quad \mathbf{J}(\phi) \wedge \psi = \phi \wedge \mathbf{J}(\psi)\,.} \tag{D.3.11}$$

dualityJ

Then, we also have self-adjointness with respect to the 6-metric,

$$\boldsymbol{\varepsilon}(\varphi,\, {}^{\#}\omega) = \boldsymbol{\varepsilon}({}^{\#}\varphi,\, \omega). \tag{D.3.12}$$

selfadj

By means of (D.3.11) and (D.2.10), electric-magnetic reciprocity and symmetry of the linear ansatz eventually lead to the spacetime relation and its inverse, namely

$$H = \lambda \, {}^{\#}F \qquad \text{and} \qquad F = -\frac{1}{\lambda} \, {}^{\#}H \, , \tag{D.3.13}$$ collectHF

which will be investigated in the next chapter.

We can consider the duality operator $^{\#}$ also from another point of view. It is our desire to describe eventually empty spacetime with such a linear ansatz. Therefore we have to reduce the number of independent functions $\overset{\text{o}}{\chi}{}^{ijkl}$ somehow. The only constants with even parity are the Kronecker deltas δ_i^j. Obviously the δ_i^j's are of no help here in specifying the $\overset{\text{o}}{\chi}{}^{ijkl}$'s, since they carry also lower indices which cannot be absorbed in a nontrivial way in order to create the 4 upper indices of $\overset{\text{o}}{\chi}{}^{ijkl}$. Recognizing that in the framework of electrodynamics in matter a similar linear ansatz can describe anisotropic media, we need a condition in order to guarantee isotropy. A "square" of $\overset{\text{o}}{\chi}$ will do the job,

$$\frac{1}{8} \; \overset{\text{o}}{\chi}{}^{klrs} \; \overset{\text{o}}{\chi}{}^{mnpq} \, \hat{\epsilon}_{rsmn} \, \hat{\epsilon}_{pqij} = -\, \delta_{ij}^{kl} \, , \tag{D.3.14}$$ close3

with δ_{rs}^{ij} as a generalized Kronecker delta. This equation, together the linearity ansatz and the symmetry, represents our fifth axiom for spacetime. Alternatively, it can also be written as

$$\frac{1}{2} \, \#_{ij}{}^{mn} \, \#_{mn}{}^{kl} = -\, \delta_{ij}^{kl} \, . \tag{D.3.15}$$ close4

D.3.3 $^{\otimes}$Algebraic solution of the closure and symmetry relations

In addition to the almost complex structure **J**, we found the symmetry of the constitutive matrix (D.3.5). As a consequence, the constraints (D.2.13) to (D.2.16), follwoing from the closure relation, pick up the additional properties

$$A = A^{\mathrm{T}}, \qquad B = B^{\mathrm{T}}, \qquad D = C^{\mathrm{T}}. \tag{D.3.16}$$ algsym

Then they reduce to

$$A^{ac}B_{cb} + C^a{}_c C^c{}_b = -\,\delta^a_b \quad \text{or} \quad AB + C^2 = -\mathbf{1}_3, \tag{D.3.17}$$ alg1

$$C^{(a}{}_c A^{b)c} = 0 \quad \text{or} \quad CA + AC^{\mathrm{T}} = 0, \tag{D.3.18}$$ alg2

$$C^c{}_{(a} B_{b)c} = 0 \quad \text{or} \quad BC + C^{\mathrm{T}}B = 0. \tag{D.3.19}$$ alg3

Naturally, we would like to resolve these algebraic constraints.

Preliminary analysis

Being a solution of the system (D.3.16)-(D.3.19), the matrix C has very specific properties. First of all, because of $\mathrm{Tr}\,\widetilde{\boldsymbol{\kappa}} = 0$, we have $\mathrm{Tr}\,C + \mathrm{Tr}\,D = 0$. With the symmetry assumption, $D = C^{\mathrm{T}}$. Thus, $\mathrm{Tr}C = 0$. More generally, the traces of all odd powers of the matrix C are zero:

$$\mathrm{Tr}C = \mathrm{Tr}(C^3) = \mathrm{Tr}(C^5) = \cdots = 0. \tag{D.3.20}$$ traceC

Indeed, multiplying (D.3.17) by C and taking the trace, we find $\mathrm{Tr}(ABC) + \mathrm{Tr}(C^3) = 0$. On the other hand, if we transpose (D.3.17) and multiply the result by C^{T}, then the trace yields $\mathrm{Tr}(BAC^{\mathrm{T}}) + \mathrm{Tr}(C^3) = 0$. The sum of the two last equations reads $\mathrm{Tr}\,\big(A(BC + C^{\mathrm{T}}B)\big) + 2\mathrm{Tr}(C^3) = 0$. In view of (D.3.19), we then conclude that $\mathrm{Tr}(C^3) = 0$. The same line of arguments yields generalizations to the higher odd powers.

It follows from (D.3.20) that the matrix C is always degenerate,

$$\det C = 0. \tag{D.3.21}$$ detC

Indeed, recall that the determinant of an arbitary 3×3 matrix $M\ (= M_b{}^a)$ reads

$$\begin{aligned} \det M &= \frac{1}{6}\,\hat{\epsilon}_{abc}\,\epsilon^{a'b'c'}\,M_{a'}{}^a M_{b'}{}^b M_{c'}{}^c \\ &= \frac{1}{6}\left[(\mathrm{Tr}M)^3 - 3\mathrm{Tr}M\,\mathrm{Tr}(M^2) + 2\mathrm{Tr}(M^3)\right]. \end{aligned} \tag{D.3.22}$$ detdef

Because of (D.3.20), we can immediately read off (D.3.21).

Let us now analyse the determinants of A and B. When the term C^2 is moved from the left-hand side of (D.3.17) to the right-hand side, a direct computation of the determinant yields:

$$\det A \, \det B = -\det(1 + C^2) = -\left(1 + \frac{\mathrm{Tr}(C^2)}{2}\right)^2 . \tag{D.3.23}$$ detABC

We used the formula (D.3.22) and the properties (D.3.20) and (D.3.21) to evaluate the right-hand side.

Accordingly, the matrices A and B cannot be both positive definite. Moreover, when at least one of them is degenerate, we find that necessarily $\mathrm{Tr}(C^2) = -2$.

General regular solution

Let us consider the case when both A and B are regular matrices, i.e., $\det A \neq 0$, $\det B \neq 0$. The general solution has been given in (D.2.27) and (D.2.28). Together with the symmetries (D.3.16), the general solution of (D.3.16) to (D.3.19) can be presented in one of the following two equivalent forms.

B-representation:

$$A = -\left[1 + (B^{-1}K)^2\right] B^{-1}, \tag{D.3.24}$$ BrepA

$$C = B^{-1}K, \tag{D.3.25}$$ BrepC

where $K = -K^{\mathrm{T}}$. The two arbitrary matrices B (symmetric) and K (antisymmetric) describe the 6+3=9 degrees of freedom of the general solution.

A-representation:

$$B = -A^{-1}\left[1 + (\widehat{K}A^{-1})^2\right], \tag{D.3.26}$$ ArepB

$$C = \widehat{K}A^{-1}, \tag{D.3.27}$$ ArepC

where $\widehat{K} = -\widehat{K}^{\mathrm{T}}$. In this case, the 6+3=9 degrees of freedom of the general solution are encoded in the matrices A (symmetric) and $\widehat{K}$ (antisymmetric).

The transition between the two representations is established with the help of the relation

$$\widehat{K} = B^{-1}KA. \qquad \text{(D.3.28)}$$ KK

One can readily check that (D.3.24), (D.3.25) and (D.3.26), (D.3.27) are really the solution of the closure and symmetry relations. Indeed, since the matrix B is non-degenerate, we find that the ansatz $C = B^{-1}K$ solves (D.3.19) provided $K + K^{\mathrm{T}} = 0$. Then from (D.3.17) we obtain the matrix A in the form (D.3.24). Finally, from (D.3.24), (D.3.25) we get

$$CA = -B^{-1}KB^{-1} - B^{-1}KB^{-1}KB^{-1}KB^{-1}. \qquad \text{(D.3.29)}$$

The right-hand side is obviously antisymmetric (i.e., the sign is changed under transposition). Hence equation (D.3.18) is satisfied identically.

If, instead, we start from a non-degenerate A, then the analogous ansatz $C = \widehat{K}A^{-1}$ solves (D.3.18), whereas B, because of (D.3.17), is found to be in the form of (D.3.26). This time, equation (D.3.19) is fullfilled because of (D.3.26) and (D.3.27).

In the case when both A and B are non-degenerate, the formulas (D.3.24), (D.3.25) and (D.3.26), (D.3.27) are merely two alternative representations of the same solution. By using (D.3.28), one can recast (D.3.24) and (D.3.25) into (D.3.26) and (D.3.27), and vice versa. However, if $\det A = 0$ and $\det B \neq 0$, then the B-representation (D.3.24), (D.3.25) can be used for the solution of the problem. In the opposite case, i.e., for $\det A \neq 0$ and $\det B = 0$, we turn to the A-representation (D.3.26) and (D.3.27). In these cases the equivalence of both sets, mediated via (D.3.28), is removed. The totally degenerate case will be treated in the next subsection.

It will be useful to write the regular solution exlicitly in components. As usual, we denote the components of the matrices as $B = B_{ab}$, $A = A^{ab}$, $C = C^a{}_b$, and the components of the *inverse* matrices as $(B^{-1}) = B^{ab}$ and $(A^{-1}) = A_{ab}$. We introduce the antisymmetric matrices by $K = K_{ab}$ and $\widehat{K} = \widehat{K}^{ab}$. Then the component version of the B-representation (D.3.24), (D.3.25) reads:

$$
\begin{aligned}
A^{ab} &= -B^{ab} - B^{am} K_{mc} B^{cd} K_{dn} B^{nb} \\
&= -B^{ab} + \frac{1}{\det B}\,(k^2\, B^{ab} - k^a\, k^b), \qquad \text{(D.3.30)} \quad \texttt{Aab} \\
C^a{}_b &= B^{ac} K_{cb} \\
&= B^{ac}\, \hat{\epsilon}_{cbd}\, k^d = \frac{1}{\det B}\, \epsilon^{acd}\, B_{cb}\, k_d. \qquad \text{(D.3.31)} \quad \texttt{Cab}
\end{aligned}
$$

Here we introduced $k^a := \frac{1}{2}\epsilon^{abc} K_{bc}$ and $k_a := B_{ab}\, k^b$, moreover, $k^2 := k_a k^a$. Analogously, the A-representation (D.3.26), (D.3.27) reads:

$$
\begin{aligned}
B_{ab} &= -A_{ab} - A_{am} \widehat{K}^{mc} A_{cd} \widehat{K}^{dn} A_{nb} \\
&= -A_{ab} + \frac{1}{\det A}\,(\widehat{k}^2\, A_{ab} - \widehat{k}_a\, \widehat{k}_b), \qquad \text{(D.3.32)} \quad \texttt{Bab} \\
C^a{}_b &= \widehat{K}^{ac} A_{cb} \\
&= A_{bc}\, \epsilon^{acd}\, \widehat{k}_d = \frac{1}{\det A}\, \hat{\epsilon}_{cbd}\, A^{ac}\, \widehat{k}^d, \qquad \text{(D.3.33)} \quad \texttt{Cab1}
\end{aligned}
$$

where $\widehat{k}_a := \frac{1}{2}\hat{\epsilon}_{abc} \widehat{K}^{bc}$, $\widehat{k}^a := A^{ab}\, \widehat{k}_b$, and $\widehat{k}^2 := \widehat{k}_a\, \widehat{k}^a$.

Degenerate solution

Besides the regular case of above, the closure and symmetry relations also admit a degenerate case when all the matrices are singular, i.e.

$$
\det A = \det B = 0. \qquad \text{(D.3.34)} \quad \texttt{zeroABdet}
$$

Recall that we always have $\det C = 0$, see (D.3.21).

We will not give here a detailed analysis of the degenerate case because, in a certain sense to be explained below, it reduces to the regular solution. Nevertheless, let us outline the main steps which yield the explicit construction of the degenerate solution. The basic tool for this will be the use of the "gauge" freedom of the system (D.3.16)-(D.3.19) which is obviously invariant under the action of the general linear group $GL(3,\mathbb{R}) \ni L_b{}^a$ defined by

$$\begin{aligned} A^{ab} &\longrightarrow L_c{}^a L_d{}^b\, A^{cd}, \\ B_{ab} &\longrightarrow (L^{-1})_a{}^c (L^{-1})_b{}^d\, B_{cd}, \\ C^a{}_b &\longrightarrow L_c{}^a (L^{-1})_b{}^d\, C^c{}_d. \end{aligned} \tag{D.3.35}$$

gl3ABC

This transformation does not change the determinants of the matrices and hence, by means of (D.3.35), the degenerate solutions are mapped again into degenerate ones. We can use the freedom (D.3.35) in order to simplify the construction of the singular solutions.

The vanishing of a determinant $\det B = 0$ means that the algebraic rank of the matrix B is less than 3 (and the same for A). A rather lengthly analysis then shows that the system (D.3.16) to (D.3.19) does not have real solutions when the matrices A or B have rank 2. As a result, we have to admit that both, A and B, carry rank 1. Then direct inspection shows that the degenerate matrices A and B can be represented in the general form

$$A^{ab} = -v^a v^b, \qquad B_{ab} = u_a u_b. \tag{D.3.36}$$

ABdeg

We substitute (D.3.36) into (D.3.17). This yields, for the square of the C matrix, $C^a{}_c C^c{}_b = -\delta^a_b + (v^c u_c)\, v^a u_b$. Taking into account the constraint $\mathrm{Tr}(C^2) = -2$, which arises from (D.3.23), we find the general structure of C^2 as

$$C^a{}_c C^c{}_b = -\delta^a_b + v^a u_b, \qquad \text{with} \qquad v^c u_c = 1. \tag{D.3.37}$$

Csquare

If we multiply $(\text{D.3.37})_1$ with u_a and v^b, respectively, we find that u_a and v^b are eigenvectors of C^2 with eigenvalues zero; this is necessary for the validity of the equations (D.3.18) and (D.3.19).

It remains to find the matrix C as the square root of $(\text{D.3.37})_1$. Although this is a rather tedious task, one can solve it with the help of the linear transformations (D.3.35). It is always possible to use (D.3.35) and to bring the column v^a and the row u_a into the specific form

$$\overset{\circ}{v}{}^a = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \overset{\circ}{u}_a = (0,0,1)\,. \tag{D.3.38}$$

va0

Then (D.3.37) can be solved explicitly and yields

$$\overset{\circ}{C}{}^a{}_b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{D.3.39}$$

Cdeg

Summarizing, the general degenerate solution is given by the matrices A, B of (D.3.36), with $v^a = L_b{}^a\,\overset{\circ}{v}{}^b$, $u_a = (L^{-1})_a{}^b\,\overset{\circ}{u}_b$, and the matrix $C^a{}_b = L_c{}^a(L^{-1})_b{}^d\,\overset{\circ}{C}{}^c{}_d$. An arbitrary matrix $L_b{}^a \in GL(3,\mathbb{R})$ embodies the 9 degrees of freedom of this solution.

D.3.4 From a quartic wave surface to the lightcone

After closure and symmetry have been taken care of in the last section and the explicit form of the matrix **J** been determined, we can come back to the Fresnel equation (D.1.55) and its M coefficients (D.1.60) to (D.1.64). The latter can now be calculated. The regular and the degenerate solutions should be considered separately. Let us begin with the regular case.

Regular case

Starting from (D.3.30)-(D.3.31) or from (D.3.32)-(D.3.33), direct calculation yields for the coefficients of the Fresnel equation

(D.1.60)-(D.1.64):

$$M = -\frac{1}{\det B}\left(1-\frac{k^2}{\det B}\right)^2 \tag{D.3.40}$$ rB1

$$= \det A, \tag{D.3.41}$$ rA1

$$M^a = \frac{1}{\det B}\,4k^a\left(1-\frac{k^2}{\det B}\right) \tag{D.3.42}$$ rB2

$$= 4\widehat{k}^a, \tag{D.3.43}$$ rA2

$$M^{ab} = -\frac{1}{\det B}\,4k^a k^b + 2B^{ab}\left(1-\frac{k^2}{\det B}\right) \tag{D.3.44}$$ rB3

$$= -2A^{ab} + \frac{6}{\det A}\,\widehat{k}^a\widehat{k}^b, \tag{D.3.45}$$ rA3

$$M^{abc} = -\,4\,B^{(ab}\,k^{c)} \tag{D.3.46}$$ rB4

$$= \frac{4}{\det A}\left(-\,A^{(ab}\widehat{k}^{c)} + \frac{1}{\det A}\widehat{k}^a\widehat{k}^b\widehat{k}^c\right), \tag{D.3.47}$$ rA4

$$M^{abcd} = -\,(\det B)\,B^{(ab}B^{cd)} \tag{D.3.48}$$ rB5

$$= \frac{1}{\det A}\left(A^{(ab}A^{cd)} - \frac{2A^{(ab}\widehat{k}^c\widehat{k}^{d)}}{\det A} + \frac{\widehat{k}^a\widehat{k}^b\widehat{k}^c\widehat{k}^d}{\det A}\right). \tag{D.3.49}$$ rA5

Here every M is described by two lines, the first one displaying the expression in terms of the B-representation whereas the second line refer to the A-representation. Substituting all this into the general Fresnel equation (D.1.55), we find

$$\begin{aligned}\mathcal{W} = &-\frac{q_0^2}{\det B}\left[q_0^2\left(1-\frac{k^2}{\det B}\right) - 2q_0q_ak^a - q_aq_b(\det B)B^{ab}\right]^2 \\ = &\;\frac{q_0^2}{\det A}\left[q_0^2\det A + 2q_0q_a\widehat{k}^a - q_aq_b\left(A^{ab} - \frac{\widehat{k}^a\widehat{k}^b}{\det A}\right)\right]^2 \\ = &-q_0^2\left(q_iq_j\,g^{ij}\right)^2 = 0.\end{aligned} \tag{D.3.50}$$ Wgij

Here g^{ij} is the symmetric tensor field. Let, for definitness, $\det B > 0$. Hence, $\det A < 0$. Then from (D.3.50) we read off the components

$$g^{ij} = \frac{1}{\sqrt{\det B}} \left(\begin{array}{c|c} 1-(\det B)^{-1}\, B_{cd}\, k^c\, k^d & -\,k^b \\ \hline -\,k^a & -\,(\det B)\, B^{ab} \end{array}\right) \tag{D.3.51}$$ gijBup

$$= \frac{1}{\sqrt{-\det A}} \left(\begin{array}{c|c} \det A & \widehat{k}^b \\ \hline \widehat{k}^a & -A^{ab} + (\det A)^{-1}\widehat{k}^a\widehat{k}^b \end{array}\right) \tag{D.3.52}$$ gijAup

One can prove that this tensor has Lorentz signature. Hence it can be understood as the metric of spacetime.

Thus, from our general analysis, we indeed recover the null or light-cone structure $q_i q^i = q_i q_j\, g^{ij} = 0$ for the propagation of electromagnetic waves: Provided the linear spacetime relation satisfies closure and symmetry, the quartic surface in (D.1.55) reduces to the lightcone for the metric g^{ij}. A visualization of the corresponding conformal manifold is given in Fig. D.3.4.

Degenerate case

The same conclusion is also true for the degenerate case. Substituting (D.3.36) into (D.1.61)-(D.1.64), and using (D.3.37)-(D.3.39), we find

$$M = 0, \qquad M^a = 0, \tag{D.3.53}$$

$$M^{ab} = -4v^a v^b, \tag{D.3.54}$$

$$M^{abc} = 4u_e C^{(a}{}_f v^b \epsilon^{c)ef}, \tag{D.3.55}$$

$$M^{abcd} = -u_e C^{(a}{}_f \epsilon^{b|ef} u_g C^{|c}{}_h \epsilon^{d)gh}. \tag{D.3.56}$$

Inserting this into (D.1.55), we obtain

$$\mathcal{W} = -q_0^2 \left[-2q_0 q_a v^a + q_a q_b u_e C^a{}_f \epsilon^{bef}\right]^2 = -q_0^2 \left(q_i q_j\, g^{ij}\right)^2 = 0. \tag{D.3.57}$$

This time the tensor field g^{ij} is described by

$$g^{ij} = \left(\begin{array}{c|c} 0 & -\,v^b \\ \hline -\,v^a & u_c C^{(a}{}_d \epsilon^{b)cd} \end{array}\right). \tag{D.3.58}$$ gijDEG

This tensor is nondegenerate and it has Lorentzian signature.

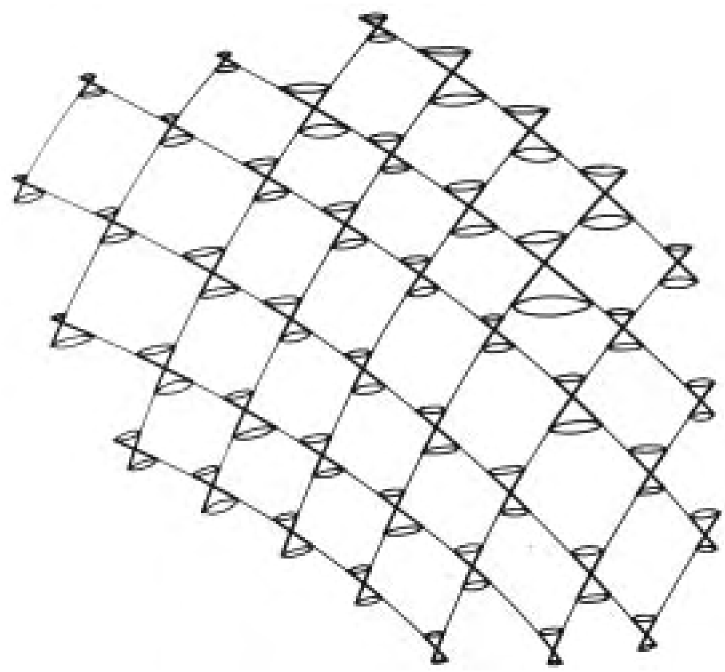

Figure D.3.1: Null cones fitted together to form a conformal manifold, see Pirani and Schild [25].

Is closure also necessary for the lightcone?

The closure property is sufficient for the lightcone to exist. Is it also a necessary condition for this?

We can demonstrate some evidence in favor of the conjecture that the closure relation is not only a sufficient, but also a necessary condition for the reduction of the quartic geometry (D.1.55) to the lightcone. For the specific case when the matrix $C = 0$, we will be able to prove also necessity.

Putting $C^a{}_b = 0$, we find from (D.1.61)-(D.1.64) that $M^a = 0$ and $M^{abc} = 0$, whereas

$$\begin{aligned} M^{ab} &= B_{cd}(A^{ab}\,A^{cd} - A^{ac}\,A^{bd}), && \text{(D.3.59)} \\ M^{abcd} &= (\det B)\,A^{(ab}\,B^{cd)}. && \text{(D.3.60)} \end{aligned}$$

Consequently, (D.1.55) reduces to

$$\mathcal{W} = q_0^2\left(\det A\, q_0^4 + q_0^2\,\gamma + \det B\,\alpha\,\beta\right) = 0, \qquad \text{(D.3.61)}$$

where $\alpha := A^{ab}q_aq_b$, $\beta := B^{ab}q_aq_b$, and $\gamma := M^{ab}q_aq_b$. Assuming that the last equation describes a lightcone, one concludes that the roots for q_0^2 should coincide. Thus necessarily

$$\gamma^2 = 4\det A\det B\,\alpha\,\beta. \qquad \text{(D.3.62)}$$ abg

Let us write $(\det A\det B) = s\,|\det A\det B|$, with $s = \mathrm{sign}(\det A \det B)$. Then (D.3.62) yields

$$2\sqrt{|\det A\det B|}\,\frac{\alpha}{\gamma} = s\,\lambda, \qquad 2\sqrt{|\det A\det B|}\,\frac{\beta}{\gamma} = \frac{1}{\lambda}, \qquad \text{(D.3.63)}$$

where λ is an arbitrary scalar factor. Recalling the definitions of α, β, γ, we then find

$$A^{ab} = s\,\lambda^2\,B^{ab}. \qquad \text{(D.3.64)}$$ AB

Consequently, $M = \det A = s\,\lambda^6/\det B$ and $M^{ab} = 2\lambda^4 B^{ab}$. Therefore one verifies that

$$\mathcal{W} = \frac{s\,\lambda^2 q_0^2}{\det B}\left(\lambda^2 q_0^2 + s\,q_aq_b\,B^{ab}\det B\right)^2 = 0. \qquad \text{(D.3.65)}$$ quad

For $s = -1$, we immediately recognize that the quadratic form in (D.3.65) can have either the $(+ - --)$ signature, or $(+ + +-)$. Similarly, for $s = 1$, the signature is either $(+ + ++)$, or $(+ + --)$. Therefore, the Fresnel equation describes the correct (hyperbolic) light*cone* structure only in the case $s = -1$.

Finally, one can verify that the above solutions satisfies

$$\kappa_{ij}{}^{mn}\kappa_{mn}{}^{kl} = s\lambda^2\delta_{ij}^{kl}, \tag{D.3.66}$$

which reproduces the closure relation $(D.2.6)_1$ for $s = -1$.

D.4

Extracting the conformally invariant part of the metric by an alternative method

The discussion above of the wave propagation in linear electrodynamics shows that the closure and symmetry relations enable us to obtain the *spacetime metric* from the spacetime relation.

Here we will present an alternative construction of the Lorentzian metric from the constitutive coefficients. As above, the crucial point will be a well-known mathematical fact. Here it is the one-to-one correspondence between the duality operators $^{\#}$ and the conformal classes of the metrics of spacetime. Hence, if we take as fifth axiom the linearity condition (D.1.6) together with closure (D.2.6) and symmetry (D.3.5), then we can construct, up to a conformal factor, the metric of spacetime from the components of the duality operator $^{\#}$. In this sense, the metric is a derived concept from the electrodynamic spacetime relation.

At the center of this derivation is the triplet of (anti)self-dual 2-forms which provides the basis of the (anti)self-dual subspace of the complexified space of all 2-forms $M^6(\mathbb{C})$.

D.4.1 $^{\otimes}$Triplet of self-dual 2-forms and metric

In Sec. C.2.5 we saw that the basis of the self-dual 2-forms can be described either by (C.2.45) or by (C.2.46). These triplets are linearly dependent, and one can use any of them for the actual computation. For example, one can take as the fundamental triplet $S^{(a)} = \overset{(s)}{\beta}{}^{a}$, with $G^{ab} = A^{ab}/4$. Alternatively, we can work with $S^{(a)} = B^{ab}\, \overset{(s)}{\epsilon}{}_{b}$, with $G^{ab} = B^{ab}/4$. Both choices yield equivalent results if A and B are non-degenerate. For definiteness, let us choose the second option. Then from (C.2.46), we have in the B-representation explicitly

$$\begin{aligned} S^{(a)} &= \frac{1}{2}\, S^{(a)}_{ij}\, dx^i \wedge dx^j = B^{ab}\, \overset{(s)}{\epsilon}{}_{b} \\ &= \frac{1}{2}\Big(- i\, dx^0 \wedge dx^a + i(\det B)^{-1}\, k_b\, dx^b \wedge dx^a \\ &\qquad + \frac{1}{2} B^{ab}\, \hat{\epsilon}_{bcd}\, dx^c \wedge dx^d \Big). \end{aligned} \tag{D.4.1}$$

Sforms1

Here $x^0 = \sigma$ and x^a are the spatial coordinates, with $a, b, c, \dots = 1, 2, 3$.

A useful calculational tool is provided by a set of *1-forms* $S^{(a)}_i := \partial_i \lrcorner\, S^{(a)} = S^{(a)}_{ij} dx^j$. With their help, we can read off the contractions needed in the Schönberg-Urbantke formulas (C.2.73)-(C.2.74) from the exterior products

$$\begin{aligned} S^{(a)}_i \wedge S^{(b)} \wedge S^{(c)}_j &= \frac{1}{2}\, S^{(a)}_{ik} S^{(b)}_{lm} S^{(c)}_{nj}\, dx^k \wedge dx^l \wedge dx^m \wedge dx^n \\ &= \frac{1}{2}\, \epsilon^{klmn}\, S^{(a)}_{ik} S^{(b)}_{lm} S^{(c)}_{nj}\, \mathrm{Vol}. \end{aligned} \tag{D.4.2}$$

SwSwS

Here, as usual, $\mathrm{Vol} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$.

From (D.4.1) we have

$$S^{(a)}_0 = -\frac{i}{2}\, dx^a, \tag{D.4.3}$$

$$\begin{aligned} S^{(a)}_b &= -\frac{i}{2}\Big(- dx^0\, \delta^a_b + e\, B^{ac}\, \hat{\epsilon}_{cbd}\, dx^d \\ &\qquad + (\det B)^{-1}\, k_c\, dx^c\, \delta^a_b - (\det B)^{-1}\, k_b\, dx^a \Big). \end{aligned} \tag{D.4.4}$$

By a rather lengthy calculation we find

$$G_{ab}\, S^{(a)} \wedge S^{(b)} = -\,6i\,\mathrm{Vol}, \tag{D.4.5}$$ det

$$\hat{\epsilon}_{abc}\, S_0^{(a)} \wedge S^{(b)} \wedge S_0^{(c)} = -\,\frac{3i}{4}\,\mathrm{Vol}, \tag{D.4.6}$$ S00

$$\hat{\epsilon}_{abc}\, S_n^{(a)} \wedge S^{(b)} \wedge S_0^{(c)} = \frac{3i}{4}\,(\det B)^{-1}\, k_n\,\mathrm{Vol}, \tag{D.4.7}$$ Sn0

$$\hat{\epsilon}_{abc}\, S_m^{(a)} \wedge S^{(b)} \wedge S_n^{(c)} = -\,\frac{3i}{4}\Big(-\,(\det B)^{-1}\, B_{mn} + (\det B)^{-2}\, k_m\, k_n\Big)\,\mathrm{Vol}. \tag{D.4.8}$$ Smn

We use (D.4.5)-(D.4.8) together with (D.4.2). Then, the Schönberg-Urbantke formulas (C.2.73)-(C.2.74), for the components of the spacetime metric, yield

$$g_{ij} = \frac{1}{\sqrt{\det B}} \left(\begin{array}{c|c} \det B & -\,k_b \\ \hline -\,k_a & -\,B_{ab} + (\det B)^{-1}\, k_a\, k_b \end{array}\right). \tag{D.4.9}$$ gijB

One can find the determinant of this expression and verify that $(i\sqrt{\det g}) = \sqrt{-\det g} = 1$, in accordance with (C.2.74).

If, instead of the B-representation (D.3.30), (D.3.31) of the solution of the closure and symmetry relations, we start from the A-representation (D.3.32), (D.3.33), then the Schönberg-Urbantke formulas yield an alternative form of the spacetime metric,

$$g_{ij} = \frac{1}{\sqrt{-\det A}} \left(\begin{array}{c|c} 1 - (\det A)^{-1}\, A^{cd}\widehat{k}_c\widehat{k}_d & -\,\widehat{k}_b \\ \hline -\,\widehat{k}_a & -\,(\det A)\, A_{ab} \end{array}\right). \tag{D.4.10}$$ gijA

The triplet of the self-dual 2-forms $S^{(a)}$ is defined up to an arbitrary scalar factor: By multiplying them with an arbitrary (in general complex) function $h(x)$, one preserves the completeness condition (C.2.52), (C.2.53). Correspondingly, the determinant of the metric will be rescaled by a factor h^4, whereas the metric itself will be rescaled by a factor h. In other words, the whole procedure defines a *conformal class* of metric rather than a metric itself. Clearly, one can always choose the conformal factor h such as to eliminate the first factor in (D.4.9).

For the metric (D.4.9), the spacetime interval explicitly reads:

$$ds^2 = \frac{1}{\sqrt{\det B}} \left[\det B \left(d\sigma - \frac{k_a dx^a}{\det B} \right)^2 - B_{ab}\, dx^a dx^b \right] . \tag{D.4.11}$$

gij1

However, as just discussed, a conformal factor is irrelevant. Therefore we may limit ourselves to the interval

$$ds'^2 = \det B \left(d\sigma - \frac{k_a dx^a}{\det B} \right)^2 - B_{ab}\, dx^a dx^b . \tag{D.4.12}$$

Yur2

If we consider the 3×3 matrix B, we can distinguish four different cases with the signatures $(+++)$, $(-++)$, $(--+)$, and $(---)$, respectively. Let us demonstrate that all these cases lead to a Lorentzian signature:

For $(+++)$, the determinant becomes positive and the expression $B_{ab}\, dx^a dx^b$ is positive definite. Thus, we can read off the Lorentzian signature immediately: σ will be the time coordinate, the x^a's the three spatial coordinates.

For $(-++)$, the determinant becomes negative and the expression $B_{ab}\, dx^a dx^b$ is indefinite. Therefore the square of one dx coordinate differential, say $(dx^1)^2$, carries a positive sign and can be identified as the time coordinate, whereas σ can be identified as space coordinate.

For $(--+)$, the determinant becomes positive again and the expression $-B_{ab}\, dx^a dx^b$ is indefinite with signature $(++-)$. Therefore x^3 is the time coordinate in this case.

For $(---)$, the determinant becomes negative and the expression $-B_{ab}\, dx^a dx^b$ is positive definite. In other words, this corresponds to the case $(+++)$ with an overall sign change, q.e.d..

Accordingly, the structure in (D.4.12) is rather robust and the quantity k_a doesn't influence the Lorentzian signature of (D.4.12).

It is quite satisfactory to note that the Schönberg-Urbantke formalism produces the same Lorentzian metric that we recovered earlier in Sec. D.3.4 when discussing the reduction of the fourth order wave surface to the lightcone. In that approach, since the wave vector is a 1-form, we obtained the contravariant components (D.3.51) and (D.3.52) of the metric which are just the inverse to (D.4.9) and (D.4.10), respectively. However, our new "reduction" method produces a spacetime metric also for the degenerate solution of the closure relation, see (D.3.58). The Schönberg-Urbantke formulas are inapplicable to the degenerate solutions. In this sense, the reduction method is further reaching.

D.4.2 Maxwell-Lorentz spacetime relation and Minkowski spacetime

Let us take a particular example for the spacetime relation. We assume that we are in a suitable frame such that we measure

$$\begin{aligned} &H_{01} = \frac{1}{\mu_0}\,F_{23}, \quad H_{02} = \frac{1}{\mu_0}\,F_{31}, \quad H_{03} = \frac{1}{\mu_0}\,F_{12}, \\ &H_{23} = -\,\varepsilon_0\,F_{01}, \quad H_{31} = -\,\varepsilon_0\,F_{02}, \quad H_{12} = -\,\varepsilon_0\,F_{03}. \end{aligned} \tag{D.4.13}$$

exCR

This set is known as the *Maxwell-Lorentz* spacetime relation in the frame chosen. The constitutive coefficients (D.1.6), for the example discussed, are independent of the spacetime coordinates and are given by (D.4.13) in terms of a pair of fundamental constants ε_0, μ_0, with $(a, b, c = 1, 2, 3)$

$$[\varepsilon_0] = \left[\frac{H_{ab}}{F_{0c}}\right] = \left[\frac{H/l^2}{F/(tl)}\right] = \frac{q^2}{h}\frac{l}{t} \overset{\text{SI}}{=} \frac{Am}{Vs} = \frac{1}{\Omega}\,\frac{m}{s}\,, \tag{D.4.14}$$

epsmu'

$$[\mu_0] = \left[\frac{F_{ab}}{H_{0c}}\right] = \left[\frac{F/l^2}{H/(tl)}\right] = \frac{h}{q^2}\frac{l}{t} \overset{\text{SI}}{=} \frac{Vm}{As} = \Omega\,\frac{m}{s}\,. \tag{D.4.15}$$

We can read off ${\kappa_{ij}}^{kl}$ from (D.4.13). Then a direct computation shows that the quadratic invariant (D.2.7) is equal to $\lambda^2 = \varepsilon_0/\mu_0$, with $[\lambda] = q^2/h \overset{\text{SI}}{=} 1/\Omega$. Consequently, the duality operator $^\#$ of (D.3.11), as defined by the spacetime relation (D.4.13), is given by the 6×6 matrix

$$\#_I{}^K = \begin{pmatrix} 0 & -\frac{1}{c}\,\mathbf{1}_3 \\ c\,\mathbf{1}_3 & 0 \end{pmatrix}, \tag{D.4.16}$$

ex1

cf. (D.1.6). Here we denoted $c := 1/\sqrt{\varepsilon_0\mu_0}$, with $[c] = \textit{velocity}$. Thus we have the matrices $A = -B^{-1}$, $B = c\,\mathbf{1}_3$, whereas $C = 0$. One then immediately finds the metric (D.4.9), (D.4.10) in the form:

$$g_{ij} = \frac{1}{\sqrt{c}} \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{D.4.17}$$

This is the standard Minkowski metric in orthonormal coordinates.

D.4.3 Hodge star operator and isotropy

The inverse of (D.4.9) is given by

$$g^{ij} = \frac{1}{\sqrt{\det B}} \left(\begin{array}{c|c} 1 - (\det B)^{-1}k^2 & -k^b \\ \hline -k^a & -(\det B)B^{ab} \end{array} \right) . \tag{D.4.18}$$

metricinv

With the help of (D.4.9) and (D.4.18), we can define the corresponding Hodge star operator * attached to the metric extracted by means of the Schönberg-Urbantke formalism. Its action on a 2-form, say F, is described by (C.2.92), or explicitly, by

$$^\star F_{ij} := \frac{\sqrt{-g}}{2}\,\hat{\epsilon}_{ijkl}\, g^{km} g^{ln} F_{mn} . \tag{D.4.19}$$

Such a Hodge duality operator, see (C.2.33), has the spacetime relation matrix $\overset{g}{\kappa}{}_{ij}{}^{kl} = \hat{\epsilon}_{ijmn}\,\overset{g}{\chi}{}^{mnkl}/2$ with

$$\overset{g}{\chi}{}^{ijkl} = \sqrt{-g}\,\left(g^{ik}g^{jl} - g^{jk}g^{il}\right) . \tag{D.4.20}$$

Note that $\overset{g}{\chi}{}^{ijkl}$ is invariant under conformal transformations $g_{ij} \rightarrow e^{\lambda(x)}g_{ij}$; this takes care that only 9 of the possible 10 components of the metric can ever enter $\overset{g}{\chi}{}^{ijkl}$.

We can compare $\overset{g}{\chi}{}^{ijkl}$ with the original $\overset{o}{\chi}{}^{ijkl}$ of the linear spacetime relation. For this purpose, we have to substitute the metric components (D.4.9) into the A, B, C blocks (C.2.95)-(C.2.97) of the Hodge duality matrix (C.2.94). Then inspection of (D.3.30)-(D.3.33) demonstrates the exact coincidence

$$\overset{g}{A}{}^{ab} = A^{ab}, \qquad \overset{g}{B}_{ab} = B_{ab}, \qquad \overset{g}{C}{}^{a}{}_{b} = C^{a}{}_{b}. \tag{D.4.21}$$

Summing up, it turns out that $\overset{g}{\chi}{}^{IJ} = \overset{o}{\chi}{}^{IJ}$, i.e., *the metric extracted allows us to write the original duality operator* $^{\#}$ *as Hodge star operator* associated to that metric. Therefore, the original linear constitutive tensor (D.1.13) can then be finally written as

$$\chi^{ijkl} = \zeta(x)\,\sqrt{-g}\,\left(g^{ik}g^{jl} - g^{jk}g^{il}\right) + \alpha(x)\,\epsilon^{ijkl}\,. \tag{D.4.22}$$

decomp1

This representation naturally suggests to interprete $\zeta(x)$ as a scalar field of a *dilaton* type.[1]

Given a metric, we can define the notion of *local isotropy*. Let $T^{i_1 \dots i_p}$ be the contravariant coordinate components of a tensor field and $T^{\alpha_1 \dots \alpha_p} := e_{i_1}{}^{\alpha_1} \cdots e_{i_p}{}^{\alpha_p}\, T^{i_1 \dots i_p}$ its frame components with respect to an orthonormal frame $e_\alpha = e^i{}_\alpha\,\partial_i$. A tensor is said to be locally isotropic at a given point, if its frame components are invariant under a Lorentz rotation of the orthonormal frame. Similar considerations extend to tensor densities.

There are only two geometrical objects which are numerically invariant under (local) Lorentz transformations: the Minkowski metric $o^{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$ and the Levi-Civita tensor density $\epsilon^{\alpha\beta\gamma\delta}$. Thus

$$\mathcal{T}^{\alpha\beta\gamma\delta} = \phi(x)\,\sqrt{-g}\,\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\beta\gamma}g^{\alpha\delta}\right) + \varphi(x)\,\epsilon^{\alpha\beta\gamma\delta} \tag{D.4.23}$$

iso

is the most general locally isotropic contravariant fourth rank tensor density of weight +1 with the symmetries $\mathcal{T}^{\alpha\beta\gamma\delta} = -\mathcal{T}^{\beta\alpha\gamma\delta} = -\mathcal{T}^{\alpha\beta\delta\gamma} = \mathcal{T}^{\gamma\delta\alpha\beta}$. Here ϕ and φ are scalar and pseudo-scalar fields, respectively.

Accordingly, in view of (D.4.22), we have proved that the constitutive tensor (D.1.13) with the closure property (D.3.14) is *locally isotropic with respect to the metric* (D.4.9).

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[1] In the low-energy string models this factor is usually written as $\zeta(x) = e^{-b\phi(x)}$, with a constant b and the dilation field $\phi(x)$.

D.4.4 ⊗Covariance properties

The discussion above was confined to a fixed coordinate system. However, one may ask: What happens when the coordinates are changed? Or, more generally, when a local (co)frame is transformed? Up to now, we considered only a holonomic coframe $\vartheta^\alpha = \delta_i^\alpha\, dx^i$, but in physically important cases one often needs to go to nonholonomic coframes. In this section we will study the covariance properties of the Schönberg-Urbantke construction.

The behavior of the metric (C.2.73)-(C.2.74) under coordinate and frame transformations is by no means obvious. Although the fundamental completeness relation (C.2.52), (C.2.53) looks covariant, one should recall that the index ${}^{(a)}$ of the self-dual S-forms comes, in a non-covariant manner, from the $3+3$ split of the basis $\mathcal{B}^I$. A transformation of the spacetime coframe (A.1.94) acts on both types of indices in the coefficients $S^{(a)}_{\alpha\beta}$ which enter the Schönberg-Urbantke formulas (C.2.73)-(C.2.74). Thus, the determination of the new components of the metric with respect to transformed frame becomes a nontrivial problem.

For the sake of generality, we will consider an arbitrary linear transformation (A.1.94) of the coframe which includes the holonomic coordinate transformation as a particular case. With respect to the original coframe $\vartheta^\alpha = \delta_i^\alpha\, dx^i$, the duality operator is described by the components of the spacetime matrix $\#_{\alpha\beta}{}^{\mu\nu} = \delta^i_\alpha \delta^j_\beta \delta^\mu_k \delta^\nu_l\, \#_{ij}{}^{kl}$. Accordingly, the components of the extracted spacetime metric read $g_{\alpha\beta} = \delta^i_\alpha \delta^j_\beta\, g_{ij}$.

Because of the tensorial nature of the definition of the duality operator (D.1.17), a linear transformation (A.1.94) of the basis, $\vartheta^\alpha = L_{\alpha'}{}^\alpha\, \vartheta^{\alpha'}$, yields the corresponding transformation of the duality components,

$$\#_{\alpha'\beta'}{}^{\mu'\nu'} = L_{\alpha'}{}^{\alpha} L_{\beta'}{}^{\beta} L_{\mu}{}^{\mu'} L_{\nu}{}^{\nu'}\, \#_{\alpha\beta}{}^{\mu\nu}. \qquad \text{(D.4.24)}$$ dualtrafo3

Recall that $L_\alpha{}^{\alpha'}$ is the inverse of $L_{\alpha'}{}^{\alpha}$.

We will now demonstrate that the Schönberg-Urbantke construction is completely covariant and that the extracted metric

(C.2.73) transforms as

$$g'_{\alpha'\beta'} = (\det L)^{-\frac{1}{2}} \, L_{\alpha'}{}^{\alpha} L_{\beta'}{}^{\beta} \, g_{\alpha\beta} \qquad \text{(D.4.25)}$$ gijL

under the linear transformation (A.1.94), (D.4.24).

The proof is technically simple and straightforward, but it is somewhat lengthy. We have prepared the necessary tools in Sec. A.1.10. Combining now the matrix equations (A.1.97), (A.1.100), and (C.2.37), we find that, with respect to the new coframe, the duality operator

$$\# \begin{pmatrix} \beta' \\ \hat{\epsilon}' \end{pmatrix} = \begin{pmatrix} C' & A' \\ B' & C'^{T} \end{pmatrix} \begin{pmatrix} \beta' \\ \hat{\epsilon}' \end{pmatrix} \qquad \text{(D.4.26)}$$ dualB1

is described by the new matrix components

$$\begin{pmatrix} C' & A' \\ B' & C'^{T} \end{pmatrix} = \frac{1}{\det L} \begin{pmatrix} Q^{T} & W^{T} \\ Z^{T} & P^{T} \end{pmatrix} \begin{pmatrix} C & A \\ B & C^{T} \end{pmatrix} \begin{pmatrix} P & W \\ Z & Q \end{pmatrix}. \qquad \text{(D.4.27)}$$ newcoeff

This is the direct matrix remake of the tensorial version (D.4.24). The matrices P, Q, W, Z are described in (A.1.98) and (A.1.99). Explicitly, eq. (D.4.27) yields

$$A' = (\det L)^{-1} \left(Q^{T} A Q + W^{T} B W + Q^{T} C W + W^{T} C^{T} Q \right), \qquad \text{(D.4.28)}$$ A1

$$B' = (\det L)^{-1} \left(P^{T} B P + Z^{T} A Z + Z^{T} C P + P^{T} C^{T} Z \right), \qquad \text{(D.4.29)}$$ B1

$$C' = (\det L)^{-1} \left(Q^{T} C P + W^{T} C^{T} Z + Q^{T} A Z + W^{T} B P \right). \qquad \text{(D.4.30)}$$ C1

It is convenient to consider the three subcases of the linear transformation (A.1.103)-(A.1.105), because their product (A.1.102) describes a general linear tranformation.

For $L = L_1$, we have (A.1.106) with only the matrix $W^{ab} = \epsilon^{abc} \, U_c$ being nontrivial. Then (D.4.28)-(D.4.30) yields

$$\begin{aligned} A' &= A + W^{T} B W + C W + W^{T} C^{T}, \\ B' &= B, \\ C' &= C + W^{T} B. \end{aligned} \qquad \text{(D.4.31)}$$ ABC1

Comparing this with the B-representation (D.3.24) and (D.3.25), we see that the transformation (D.4.31) means a mere shift of the antisymmetric matrix:

$$K' = K + B W^{T} B, \quad \text{or, equivalently,} \quad k'_{a} = k_{a} - (\det B) \, U_{a}. \qquad \text{(D.4.32)}$$ Kcase1

Substituting this into (D.4.9), we find the transformed metric:

$$g'_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{\sqrt{\det B}} \left(\begin{array}{c|c} 0 & (\det B) \, U_{b} \\ \hline (\det B) \, U_{a} & -k_{a} U_{b} - k_{b} U_{a} + (\det B) \, U_{a} U_{b} \end{array} \right). \qquad \text{(D.4.33)}$$ gij2

Comparing with (A.1.103), we get a particular case of (D.4.25):

$$g'_{\alpha'\beta'} = L_{\alpha'}{}^{\alpha} L_{\beta'}{}^{\beta} \, g_{\alpha\beta} \qquad \text{with} \qquad L = L_1. \qquad \text{(D.4.34)}$$ gij1L

When $L = L_2$, the matrix $Z_{ab} = \hat{\epsilon}_{abc} \, V^{c}$ describes the case (A.1.107). Then, from (D.4.28)-(D.4.30), we find:

$$\begin{aligned} A' &= A, \\ B' &= B + Z^{T} A \, Z + Z^{T} C + C^{T} Z, \\ C' &= C + A Z. \end{aligned} \qquad \text{(D.4.35)}$$ ABC2

Contrary to the first case, it is now more convenient to proceed in the A-representation. From (D.3.26) and (D.3.27), we then see that the transformation (D.4.35) means a mere shift of the antisymmetric matrix

$$\widehat{K}' = \widehat{K} + AZA, \quad \text{or, equivalently,} \quad \widehat{k}'_a = \widehat{k}_a - (\det A)\, A_{ab}\, V^b. \tag{D.4.36}$$

Kcase2

Substituting this into (D.4.10), we obtain the transformed metric:

$$g'_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{\sqrt{\det A}} \left(\begin{array}{c|c} 2\widehat{k}_c\, V^c - (\det A)\, A_{cd}\, V^c V^d & (\det B)\, A_{bd} V^d \\ \hline (\det A)\, A_{ac} V^c & 0 \end{array} \right). \tag{D.4.37}$$

gij2a

We compare with (A.1.104) and recover a subcase of (D.4.25),

$$g'_{\alpha'\beta'} = L_{\alpha'}{}^{\alpha} L_{\beta'}{}^{\beta}\, g_{\alpha\beta} \qquad \text{with} \qquad L = L_2. \tag{D.4.38}$$

gij2L

Finally, for $L = L_3$ we have (A.1.108). Then (D.4.28)-(D.4.30) reduce to

$$\begin{aligned} A' &= \frac{\det \Lambda}{\Lambda_0{}^0}\, \Lambda^{-1} A (\Lambda^{-1})^T, \\ B' &= \frac{\Lambda_0{}^0}{\det \Lambda}\, \Lambda^T B \Lambda, \\ C' &= \Lambda^{-1} C \Lambda. \end{aligned} \tag{D.4.39}$$

ABC3

For the antisymmetric matrix K, this yields:

$$K' = \frac{\Lambda_0{}^0}{\det \Lambda}\, \Lambda^T K \Lambda, \quad \text{or, equivalently,} \quad k'^a = \Lambda_0{}^0\, (\Lambda^{-1})_b{}^a\, k^b. \tag{D.4.40}$$

Kcase3

As we saw above, the analysis of the case (A.1.106) was easier in the B-representation, whereas the A-representation was more suitable for the treatment of the case (A.1.107). However, the last case (A.1.108), (D.4.39) looks the same in both pictures. For definiteness, let us choose the B-representation. A new and nontrivial feature of the present case is that the transformation L_3 is not unimodular, $\det L_3 = \Lambda_0{}^0 \det \Lambda \neq 1$. Recall that $\det L_1 = \det L_2 = 1$. As a consequence, one should carefully study the behavior of the determinant of the metric defined by the second Urbantke formula (C.2.74).

From (A.1.100) we have the transformation of the self-dual basis (C.2.46):

$$\overset{(s)}{\epsilon}{}'_a = \frac{1}{\det \Lambda}\, \Lambda_a{}^b\, \overset{(s)}{\epsilon}{}_b. \tag{D.4.41}$$

Hence, for the S-forms (D.4.1), we find

$$S'^{(a)} = \frac{1}{\Lambda_0{}^0}\, (\Lambda^{-1})_b{}^a\, S^{(b)}, \tag{D.4.42}$$

and, consequently,

$$B'_{ab}\, S'^{(a)} \wedge S'^{(b)} = \frac{1}{\det L_3} B_{ab}\, S^{(a)} \wedge S^{(b)}. \tag{D.4.43}$$

Using this in (C.2.74), one finally proves the invariance of the determinant:

$$\sqrt{\det g'} = \sqrt{\det g}. \tag{D.4.44}$$

invdet

Taking this result into account when substituting (D.4.39) and (D.4.40) into (D.4.9), one obtains the transformed metric

$$g'_{\alpha\beta} = \frac{1}{\sqrt{\det B \det L_3}} \left(\begin{array}{c|c} (\Lambda_0{}^0)^2 \det B & -\Lambda_0{}^0 \Lambda_b{}^d\, k_d \\ \hline -\Lambda_0{}^0 \Lambda_a{}^c\, k_c & \Lambda_a{}^c \Lambda_b{}^d \left[-B_{cd} + (\det B)^{-1}\, k_c\, k_d \right] \end{array} \right). \tag{D.4.45}$$

gij3

Thus, the metric transforms as a tensor *density*,

$$g'_{\alpha'\beta'} = \frac{1}{\sqrt{\det L}}\, L_{\alpha'}{}^{\alpha} L_{\beta'}{}^{\beta} g_{\alpha\beta} \qquad \text{with} \qquad L = L_3. \tag{D.4.46}$$

gij3L

This transformation law is completely consistent with the invariance of the determinant (D.4.44).

Turning now to the case of a general linear transformation, one can use the factorization (A.1.102) and perform the three transformations (A.1.103)-(A.1.105) one after another. This yields subsequently (D.4.34), (D.4.38), and (D.4.46). The final result is then represented in (D.4.25) with an arbitrary transformation matrix L.

Reducing degenerate case to the regular one

Using the formalism above, we can show that the degenerate solutions of the closure relation, A, B, C, can always be transformed into the regular configurations. Indeed, let us take the degenerate solution (D.3.36), (D.3.38), (D.3.39), which is explicitly given by the three matrices

$$\overset{\circ}{A} = -\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \overset{\circ}{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \overset{\circ}{C} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{D.4.47}$$

ABCdeg

Consider a simple transformation of the coframe $\vartheta^\alpha = L_{\alpha'}{}^\alpha \, \vartheta^{\alpha'}$ by means of the $L = L_1$ matrix (A.1.103) with $U_a = (0, 0, 1)$. This induces the transformation of the 2-form basis (A.1.97) where the matrix W of (A.1.106) reads explicitly

$$W^{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{D.4.48}$$

Wdeg

The other matrices are $P = Q = \mathbf{I}_3$ and $Z = 0$. Then, using (D.4.31), we find the transformed spacetime relation matrices as

$$\overset{\circ}{A}{}' = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \overset{\circ}{B}{}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \overset{\circ}{C}{}' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{D.4.49}$$

ABCdeg1

With respect to the transformed basis, as we immediately see, the constitutive matrices become *non*-degenerate, $\det \overset{\circ}{A}{}' \neq 0$, and we can use the A-representation to describe this configuration and to construct the corresponding metric of spacetime.

Evidently, the general degenerate solution can also be reduced to the regular case. Then the linear transformation above, with $L = L_1$, should be supplemented by the appropriate $GL(3, \mathbb{R})$ transformation (D.3.35). We thus conclude that the separate treatment of the degenerate case is in fact not necessary: The degeneracy (D.3.34) merely reflects an unfortunate choice of the frame which can easily be removed by a linear transformation.

D.5

Fifth axiom

Summarizing the content of Part D, we can now give a clear formulation of the fifth axiom. In Maxwell–Lorentz electrodynamics, the 2-forms of the electromagnetic excitation H and the field strength F are related by the universal *linear* law

$$H = \boldsymbol{\kappa}(F) \qquad \text{or} \qquad H_{ij} = \frac{1}{2}\,\kappa_{ij}{}^{kl}\,F_{kl}\,, \tag{D.5.1}$$

5chiHF

with the linear operator

$$\boldsymbol{\kappa}(a\,\phi + b\,\psi) = a\,\boldsymbol{\kappa}(\phi) + b\,\boldsymbol{\kappa}(\psi)\,; \tag{D.5.2}$$

this operator fulfills *closure*

$$\boldsymbol{\kappa}^2 = -\lambda^2\,\mathbf{1}_6 \qquad \text{or} \qquad \kappa_{ij}{}^{mn}\,\kappa_{mn}{}^{kl} = -\,\lambda^2\,\delta^{kl}_{ij}, \tag{D.5.3}$$

5close

and *symmetry*

$$\boldsymbol{\kappa}(\phi)\wedge\psi = \phi\wedge\boldsymbol{\kappa}(\psi) \quad \text{or} \quad \epsilon^{ijmn}\kappa_{mn}{}^{kl} = \epsilon^{klmn}\kappa_{mn}{}^{ij}\,. \tag{D.5.4}$$

5sym

Linearity, closure,[1] and symmetry provide a unique *lightcone* structure for the propagation of electromagnetic waves, see Fig.

---

[1]With $(\text{D.5.3})_1$ and the understanding that H and F live also in the M_6, we could write $(\text{D.5.1})_1$ symbolically as $H = \lambda\sqrt{-\mathbf{1}_6}\,F$. For the square-root of such a negative unit matrix, see Gantmacher [5] pp.214 et seq.

D.3.4. As a result, the spacetime metric with the correct Lorentzian signature is, up to a conformal factor, reconstructed from the constitutive coefficients $\kappa_{ij}{}^{kl}$.

The Maxwell–Lorentz electrodynamics is specified, among other viable models, by a vanishing axion field

$$\alpha = \frac{1}{6}\,\mathrm{Tr}\,\boldsymbol{\kappa} = 0 \qquad \text{(no axion)}\,, \tag{D.5.5}$$

and a constant universal dilation factor

$$\lambda^2 = -\frac{1}{6}\,\mathrm{Tr}(\boldsymbol{\kappa}^2) = const \qquad \text{(no dilaton)}\,. \tag{D.5.6}$$

We could have excluded the axion field alternatively by insisting on electric-magnetic reciprocity of the linear law in the first place instead of only assuming closure, as in (D.5.3). Then, making use of the metric extracted and of the corresponding Hodge star operator *, the *Maxwell-Lorentz spacetime relation* can be written as

$$\boxed{H = \lambda\,{}^\star F} \qquad \text{(fifth axiom)}\,, \tag{D.5.7}$$

constvac

or, in components, as

$$H_{ij} = \frac{\lambda}{2}\,\hat{\epsilon}_{ijmn}\sqrt{-g}\,\left(g^{mk}g^{nl} - g^{nk}g^{ml}\right)F_{mn} = \lambda\hat{\epsilon}_{ij}{}^{kl}\sqrt{-g}\,F_{kl}\,. \tag{D.5.8}$$

constvac'

Here λ is a universal constant with the dimension of an impedance. Its value is

$$\lambda = \sqrt{\frac{\varepsilon_0}{\mu_0}} \approx \frac{1}{377\,\Omega}\,. \tag{D.5.9}$$

Accordingly, the experimentally well-established Maxwell-Lorentz electrodynamics is distinguished from other models by linearity, closure, symmetry, no axion, and no dilaton. Generalizations are obvious. We will discuss nonlinearity and nonlocality in Chap. E.2.

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Part E

Electrodynamics in vacuum and in matter

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E.1

Standard Maxwell–Lorentz theory in vacuum

E.1.1 Maxwell-Lorentz equations, impedance of the vacuum

When *the Maxwell-Lorentz spacetime relation* (D.5.7) is substituted into the Maxwell equations (B.4.8), (B.4.9), we find the Maxwell-Lorentz equations

$$d\,{}^\star F = J/\lambda\,, \qquad dF = 0 \tag{E.1.1}$$

MaxLor

of standard electrodynamics.

The numerical value of the constant factor (D.5.7) is fixed by experiment:

$$\lambda := \sqrt{\frac{\varepsilon_0}{\mu_0}} = \frac{e^2}{4\pi\alpha_{\rm f}\hbar} = 2.6544187283\,\frac{1}{k\,\Omega}. \tag{E.1.2}$$

lambda

Here e is the charge of the electron and $\alpha_{\rm f} = 1/137.036$ is the fine structure constant. The inverse $1/\lambda$ is called the characteristic impedance (or wave resistance) of the vacuum. This is a fundamental constant which describes the basic electromagnetic property of spacetime if considered as a special type of medium (sometimes called vacuum, or aether, in the old terminology).

In this sense, one can understand (D.5.7) as the constitutive relations for the spacetime itself. The Maxwell-Lorentz spacetime relation (D.5.7) is universal. It is equally valid in Minkowski, Riemannian, and post-Riemannian spacetimes.

The *electric constant* ε_0 and the *magnetic constant* μ_0 (also called vacuum permittivity and vacuum permeability, respectively) determine the universal constant of nature

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \tag{E.1.3}$$ cem

which gives the velocity of light in vacuum.

Making use of the homogeneous Maxwell field equation $F = dA$ and of (E.1.3), we can recast the inhomogeneous Maxwell equation $(\text{E.1.1})_1$ in the form

$$\varepsilon_0 \, d^\star F = \varepsilon_0 \, d^\star dA = \frac{1}{c} J. \tag{E.1.4}$$ linmax

Recalling the definition of the codifferential (C.2.107), $d^\dagger := {}^\star d^\star$ and of wave operator (d'Alembertian) (C.2.110), we can rewrite (E.1.4) as

$$\left(\Box - d\, d^\dagger\right) A = \frac{1}{\varepsilon_0 c} {}^\star J \,. \tag{E.1.5}$$ waveA

If, using the gauge invariance, one imposes the Lorentz gauge condition $d^\dagger A = 0$, a wave equation is found for the electromagnetic potential 1-form:

$$\Box A = \frac{1}{\varepsilon_0 c} {}^\star J \quad \text{with} \quad d^\dagger A = 0. \tag{E.1.6}$$

In components, the left-hand side reads:

$$\Box A = \left(- \widetilde{\nabla}^k \widetilde{\nabla}_k A_i + \widetilde{\text{Ric}}_i{}^k A_k \right) dx^i \,. \tag{E.1.7}$$ squareA

Here $\text{Ric}_{ij} := R_{kij}{}^k$ is the Ricci tensor, see (C.1.52). The tilde denotes the covariant differentiation and the geometric objects defined by the Levi-Civita connection (C.2.103).

Also F obeys a wave equation. We take the Hodge star of (E.1.4) and differentiate it. We substitute $dF = 0$ and find

$$\Box F = \frac{1}{\varepsilon_0 c}\, d\, {}^\star J\,. \tag{E.1.8}$$

The left hand side of this equation, in terms of components, can be determined by substituting (C.2.110) and (C.2.107):

$$\Box F = \frac{1}{2}\left(-\,\widetilde{\nabla}^k \widetilde{\nabla}_k\, F_{ij} - 2\,\widetilde{\mathrm{Ric}}_{[i}{}^{k}\, F_{j]k} + \widetilde{R}^{kl}{}_{ij}\, F_{kl}\right) dx^i \wedge dx^j\,. \tag{E.1.9}$$

DeltaF1

Accordingly, curvature dependent terms surface in a natural way both in (E.1.7) and in (E.1.9).

E.1.2 Action

According to (B.5.73), the excitation can be expressed in terms of the electromagnetic Lagrangian V by

$$H = -\frac{\partial V}{\partial F}. \tag{E.1.10}$$

fieldmom1

Because of the Maxwell-Lorentz spacetime relation (D.5.7), the excitation H is linear and homogeneous in F. Therefore the action V is homogeneous in F of degree 2. Then by Euler's theorem for homogeneous functions, we have

$$F \wedge \frac{\partial V}{\partial F} = 2V \tag{E.1.11}$$

constitmax

or, with (E.1.10) and (D.5.7),

$$V = -\frac{1}{2} F \wedge H = -\frac{\lambda}{2} F \wedge {}^\star F = -\frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\, F \wedge {}^\star F\,. \tag{E.1.12}$$

constitv

This is the twisted Lagrangian 4-form of the electromagnetic field à la Maxwell-Lorentz.

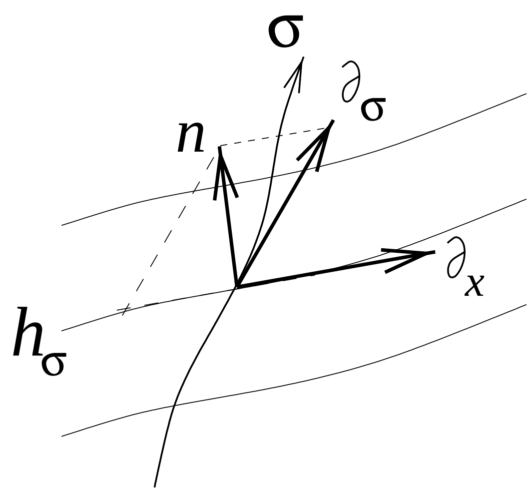

Figure E.1.1: The vector field n adapted to the (1+3)-foliation with a metric: n is orthogonal to h_σ. Compare with Fig. B.1.3.

E.1.3 Foliation of a spacetime with a metric. Effective permeabilities

In the previous sections, the standard Maxwell-Lorentz theory is presented in 4-dimensional form. In order to visualize the separate electric and magnetic pieces, we have to use the $(1+3)$ decomposition technique.

In presence of the metric, it becomes necessary to further specialize the vector field n which is our basic tool in a $(1+3)$ decomposition. Before we introduced the metric, all possible vectors n described the same spacetime foliation without really distinguishing 'time' and 'space', see Fig. B.1.3, since the very notions of *time-like* and *space-like* vectors and subspaces were absent. Now we choose the three functions n^a in such a way that

$$\mathbf{g}(n,\partial_a) = g_{(\sigma)a} + g_{ab}\, n^b = 0, \tag{E.1.13}$$ ortho

where $g_{ab} := \mathbf{g}(\partial_a,\partial_b),\ g_{(\sigma)a} := \mathbf{g}(\partial_\sigma,\partial_a),\ g_{(\sigma)(\sigma)} := \mathbf{g}(\partial_\sigma,\partial_\sigma)$. If then

$$\mathbf{g}(n,n) = g_{(\sigma)(\sigma)} - g_{ab}\, n^a n^b = N^2 > 0, \tag{E.1.14}$$ N2

the vector field n is time-like, and thus we can, indeed, consider σ as a local time coordinate. The condition (E.1.13) guarantees that the folia h_σ of constant σ are orthogonal to n, see Fig. E.1.1. Thus they are really 3-dimensional spacelike hypersurfaces. The metric

$$^{(3)}\mathbf{g}(\partial_a, \partial_b) := -\, g_{ab} \qquad \text{(E.1.15)}$$ 3met

is evidently a *positive* definite Riemannian metric on h_σ. We will denote by $\underline{*}$ the Hodge star operator defined in terms of the metric (E.1.15).

Applying the general definitions (C.2.85) to our foliation compatible coframe (B.1.31), we find the relations between 4-dimensional and 3-dimensional star operators:

$$^\star(d\sigma \wedge \overset{(p)}{\underline{\Psi}}) = (-1)^p\, \frac{1}{N}\, \underline{*}\overset{(p)}{\underline{\Psi}}, \qquad \text{(E.1.16)}$$ 34hodge1

$$^\star\overset{(p)}{\underline{\Psi}} = (-1)^p\, N\, d\sigma \wedge \underline{*}\overset{(p)}{\underline{\Psi}}. \qquad \text{(E.1.17)}$$ 34hodge2

Here $\overset{(p)}{\underline{\Psi}}$ is an arbitrary *transversal* (i.e., purely *spatial*) p-form. Note that $\underline{**} = 1$ for all forms.

Substituting the $(1+3)$ decompositions (B.2.7) and (B.1.36) into (D.5.7),

$$\underline{H} = \lambda\, \underline{({}^\star F)}, \qquad {}^\perp H = \lambda\, {}^\perp({}^\star F), \qquad \text{(E.1.18)}$$ const3a

we find the three-dimensional form of the Maxwell-Lorentz spacetime relation:

$$\mathcal{D} = \varepsilon_{\rm g}\, \varepsilon_0\, \underline{*}E \qquad \text{and} \qquad B = \mu_{\rm g}\, \mu_0\, \underline{*}\mathcal{H}\,, \qquad \text{(E.1.19)}$$ const3

where we introduced the *effective* electric and magnetic permeabilities

$$\varepsilon_{\rm g} = \mu_{\rm g} = \frac{c}{N}, \qquad \text{(E.1.20)}$$ effconst

see (D.5.7) and (E.1.3). In general, these quantities are functions of coordinates since N, according to (E.1.14), is determined by the geometry of spacetime. Thus the *gravitational field acts via its potential, the metric on spacetime and makes it look like a medium with nontrivial polarization properties.* In particular, the propagation of light, described by the Maxwell equations, is affected by these refractive properties of curved spacetime. In flat Minkowski space $N = c$, and hence $\varepsilon_{\mathrm{g}} = \mu_{\mathrm{g}} = 1$.

E.1.4 Symmetry of the energy-momentum current

If the spacetime metric g is given, then there exists a unique torsion-free and metric-compatible Levi-Civita connection $\widetilde{\Gamma}_\alpha{}^\beta$, see (C.2.103), (C.2.134). Consider the conservation law (B.5.43) of the energy-momentum. In a Riemannian space, the covariant Lie derivative $\widetilde{\mathrm{L}}_\xi = \widetilde{D}\,\xi\rfloor + \xi\rfloor\widetilde{D}$ commutes with the Hodge operator, $\widetilde{\mathrm{L}}_\xi{}^\star = {}^\star\widetilde{\mathrm{L}}_\xi$. Thus (B.5.44) straightforwardly yields

$$\widehat{X}_\alpha = \frac{\lambda}{2}\left({}^\star F \wedge \widetilde{\mathrm{L}}_{e_\alpha} F - F \wedge \widetilde{\mathrm{L}}_{e_\alpha}{}^\star F\right) = 0\,. \qquad \text{(E.1.21)}$$

Xalriem

Therefore in *general relativity* (GR), with the Maxwell-Lorentz spacetime relation, (B.5.43) simply reduces to

$$\widetilde{D}\,{}^{\mathrm{k}}\Sigma_\alpha = (e_\alpha \rfloor F) \wedge J\,. \qquad \text{(E.1.22)}$$

The energy-momentum current (B.5.8) now reads

$${}^{\mathrm{k}}\Sigma_\alpha = \frac{\lambda}{2}\left[F \wedge (e_\alpha \rfloor {}^\star F) - (e_\alpha \rfloor F) \wedge {}^\star F\right]. \qquad \text{(E.1.23)}$$

maxmomergy

In the absence of sources, $J = 0$, we find the energy-momentum law

$$\widetilde{D}\,{}^{\mathrm{k}}\Sigma_\alpha = 0\,. \qquad \text{(E.1.24)}$$

In the *flat* Minkowski spacetime of SR, we can *globally* choose the coordinates in such a way that $\widetilde{\Gamma}_\alpha{}^\beta = 0$. Thus $\widetilde{D} \stackrel{*}{=} d$ and $d\,{}^{\mathrm{k}}\Sigma_\alpha = 0$.

As we already know from (B.5.14), the current (E.1.23) is *traceless* $\vartheta^\alpha \wedge {}^{\mathrm{k}}\Sigma_\alpha = 0$. Moreover, we now can use the metric and prove also its symmetry. We multiply (E.1.23) by $\vartheta_\beta = g_{\beta\gamma}\,\vartheta^\gamma$ and antisymmetrize:

$$\frac{4}{\lambda}\,\vartheta_{[\beta}\wedge\Sigma_{\alpha]} = \vartheta_\beta\wedge F\wedge(e_\alpha\rfloor{}^\star F) - \vartheta_\beta\wedge(e_\alpha\rfloor F)\wedge{}^\star F - \vartheta_\alpha\wedge F\wedge(e_\beta\rfloor{}^\star F) + \vartheta_\alpha\wedge(e_\beta\rfloor F)\wedge{}^\star F. \tag{E.1.25}$$ momergy1

Because of (C.2.133) and (C.2.131), the first term on the right-hand side can be rewritten,

$$\vartheta_\beta\wedge F\wedge(e_\alpha\rfloor{}^\star F) = F\wedge\vartheta_\alpha\wedge{}^\star F\,(\vartheta_\beta\wedge F) = F\wedge\vartheta_\alpha\wedge(e_\beta\rfloor{}^\star F), \tag{E.1.26}$$ momergy2

i.e., it is compensated by the third term. We apply the analogous technique to the second term. Because ${}^{\star\star}F = -F$, we have

$$\vartheta_\beta\wedge{}^\star({}^\star F\wedge\vartheta_\alpha)\wedge{}^\star F = {}^\star({}^\star F\wedge\vartheta_\alpha)\wedge\vartheta_\beta\wedge{}^\star F = -{}^\star(\vartheta_\beta\wedge{}^\star F)\wedge{}^\star F\wedge\vartheta_\alpha = -\vartheta_\alpha\wedge(e_\beta\rfloor F)\wedge{}^\star F. \tag{E.1.27}$$ momergy3

In other words, the second term is compensated by the fourth one and we have

$$\vartheta_{[\beta}\wedge{}^{\rm k}\Sigma_{\alpha]} = 0. \tag{E.1.28}$$ momergy4

Alternatively, we can work with the energy-momentum tensor. We decompose the 3-form ${}^{\rm k}\Sigma_\alpha$ with respect to the η-basis. This is now possible since a metric is available. Because of $\vartheta^\alpha\wedge\eta_\gamma = \delta^\alpha_\gamma\,\eta$, we find

$${}^{\rm k}\Sigma_\alpha =: {}^{\rm k}T_\alpha{}^\beta\,\eta_\beta \qquad \text{or} \qquad {}^{\rm k}T_{\alpha\beta} = {}^\star\left(\vartheta_\beta\wedge{}^{\rm k}\Sigma_\alpha\right), \tag{E.1.29}$$ EMTensor

compare this with (B.5.28)-(B.5.29). We have

$${}^{\rm k}T_{\alpha\beta} = {}^{\rm k}\mathcal{T}_{\alpha\beta}/\sqrt{-g}. \tag{E.1.30}$$

Its tracelessness $T_\gamma{}^\gamma = 0$ has already been established, see (B.5.29), its symmetry

$${}^{\rm k}T_{[\alpha\beta]} = 0 \tag{E.1.31}$$ maxsym

can be either read off from (E.1.25) and (E.1.28) or directly from (B.5.40) with $\mathcal{H}^{ij} \sim F^{ij}$. A manifestly symmetric version of the energy-momentum tensor can be derived from (B.5.28) and (E.1.23):

$$ {}^{\mathrm{k}}T_{\alpha\beta} = -\,\lambda\,{}^{\star}\left[{}^{\star}(e_\alpha \rfloor F) \wedge (e_\beta \rfloor F) + \frac{1}{2} g_{\alpha\beta}({}^{\star}F \wedge F)\right] . \qquad \text{(E.1.32)} $$

calt

Thus ${}^{\mathrm{k}}T_{\alpha\beta}$ is a traceless symmetric tensor(-valued 0-form) with 9 independent components. Its symmetry is sometimes called a *bastard symmetry* since it interrelates two indices of totally different origin as can be seen from (E.1.28). Without using a metric, it cannot even be formulated, see (B.5.8).

It is a reflection of the symmetry of ${}^{\mathrm{k}}T_{\alpha\beta}$, that the energy flux density 2-form s (B.5.53) and the momentum density 3-form p_a (B.5.54) are closely related:

$$ p_a = -\,\frac{1}{N^2}\,\underline{dx}_a \wedge s. \qquad \text{(E.1.33)} $$

Planck

In a Minkowski space, we have $N = c$. This is the electromagnetic version of the relativistic formula $m = \frac{1}{c^2}E$ in a field-theoretic disguise.

E.2

Electromagnetic spacetime relations beyond locality and linearity

E.2.1 Keeping the first four axioms fixed

The *Lamb shift* in the spectrum of an hydrogen atom and the *Casimir force* between two uncharged conducting plates attest to the possibility to polarize spacetime electromagnetically. We have electromagnetic "vacuum polarization". These effects are to be described in the framework of quantum electrodynamics. However, since the Lamb shift and the Casimir effect are low energy effects with slowly varying electromagnetic fields involved, they can be described quasi-classically in the framework of classical electrodynamics with an altered spacetime relation. Thus the *linear* Maxwell-Lorentz relation has to be substituted by the *non*linear Heisenberg-Euler spacetime relation, see Sec. E.2.3.

It should be understood that in our axiomatic set-up, when only the Maxwell-Lorentz relation, the fifth axiom, is generalized, the fundamental structure of electrodynamics, namely the first four axioms, are untouched. If we turn the argument around: The limits of the Maxwell-Lorentz spacetime relation are visible. The fifth axiom is built on shakier grounds than the first four axioms.

Obviously, besides nonlinearity in the spacetime relation, the *nonlocality* can be and has been explored. This is further away from present-day experiments but it may be unavoidable in the end.

E.2.2 $^{\otimes}$Mashhoon

One says that a spacetime has *dispersion* properties when the electromagnetic fields produce non-instantaneous polarization and magnetization effects. The most general *linear* spacetime relation is then given, in the comoving system, by means of the Volterra integral

$$H_{ij}(\tau,\xi) = \frac{1}{2}\int d\tau' K_{ij}{}^{kl}(\tau,\tau')\, F_{kl}(\tau',\xi)\,. \qquad \text{(E.2.1)}$$ non-local

The coefficients of the kernel $K_{ij}{}^{kl}(\tau,\tau')$ are called the response functions. We expect the metric to be involved in their set-up. Their form is defined by the internal properties of spacetime itself.

Mashhoon has proposed a physically very interesting example of such a non-local electrodynamics in which non-locality comes as a direct consequence of the *non-inertial* dynamics of observers. In this case, instead of a decomposition with respect to $dx^i \wedge dx^j$, one should use the field expansions

$$H = \frac{1}{2}\, H_{\alpha\beta}\, \vartheta^\alpha \wedge \vartheta^\beta, \qquad F = \frac{1}{2}\, F_{\alpha\beta}\, \vartheta^\alpha \wedge \vartheta^\beta \qquad \text{(E.2.2)}$$ non-localHF

with respect to the coframe of a non-inertial observer $\vartheta^\alpha = e_i{}^\alpha\, dx^i$. The spacetime relation is then replaced by

$$H_{\alpha\beta}(\tau,\xi) = \frac{1}{2}\int d\tau' K_{\alpha\beta}{}^{\gamma\delta}(\tau,\tau')\, F_{\gamma\delta}(\tau',\xi)\,, \qquad \text{(E.2.3)}$$ non-local1

and the response kernel in (E.2.3) is now defined by the acceleration and rotation of the observer's reference system. It is a constitutive law for the vacuum as viewed from a non-inertial frame of reference.

Mashhoon imposes an additional *assumption* that the kernel is of *convolution* type, i.e., $K_{\alpha\beta}{}^{\gamma\delta}(\tau,\tau') = K_{\alpha\beta}{}^{\gamma\delta}(\tau-\tau')$. Then the kernel can be uniquely determined by means of the Volterra technique, and often it is possible to use the Laplace transformation in order to simplify the computations. Unfortunately, although Mashhoon's kernel is always calculable in principle, in actual practice one normally cannot obtain K explicitly in terms of the observer's acceleration and rotation.

Preserving the main ideas of Mashhoon's approach, one can abandon the convolution condition. Then the general form of the kernel can be worked out explicitly (u is the observer's 4-velocity):

$$K_{\alpha\beta}{}^{\gamma\delta}(\tau,\tau') = \frac{1}{2}\,\epsilon_{\alpha\beta}{}^{\lambda[\delta}\left(\delta_{\lambda}^{\gamma]}\,\delta(\tau-\tau') - u\rfloor\Gamma_{\lambda}{}^{\gamma]}(\tau')\right). \qquad \text{(E.2.4)}$$

NewAnsatz

The influence of non-inertiality is manifest in the presence of the connection 1-form. The kernel (E.2.4) coincides with the original Mashhoon kernel in the case of constant acceleration and rotation, but in general the two kernels are different.[1] Perhaps, only the direct observations would establish the true form of the non-local spacetime relation. However, such a non-local effect has not been confirmed experimentally as yet.

E.2.3 Heisenberg-Euler

Quantum electrodynamical vacuum corrections to the Maxwell-Lorentz theory can be accounted for by an effective *nonlinear* spacetime relation derived by Heisenberg and Euler. To the first order in the fine structure constant $\alpha_{\rm f} = \frac{e^2}{4\pi\varepsilon_0\hbar c}$, it is given by[2]

$$H = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left\{\left[1 + \frac{8\,\alpha_{\rm f}}{45\,B_{\rm k}^2}\,{}^\star\!\left(F\wedge{}^\star F\right)\right]{}^\star F + \frac{14\,\alpha_{\rm f}}{45\,B_{\rm k}^2}\,{}^\star\!\left(F\wedge F\right)F\right\}, \qquad \text{(E.2.5)}$$

HE

[1]See Muench et al. [16].

[2]See Itzykson and Zuber [12] or Heyl and Hernquist [10], e.g..

where the magnetic field strength $B_{\rm k} = \frac{m^2c^2}{e\hbar} \approx 4.4 \times 10^9$ T, with the mass of the electron m. Again, post-Riemannian structures don't interfere here. This theory is a valid physical theory.

According to (E.2.5), the vacuum is treated as a specific type of a medium which polarizability and magnetizability properties are determined by the "clouds" of virtual charges surrounding the real currents and charges.

E.2.4 $^{\otimes}$Born-Infeld

The non-linear Born-Infeld theory represents a *classical* generalization of the Maxwell-Lorentz theory for accommodating stable solutions for the description of 'electrons'. Its spacetime relation reads[3]

$$H = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{{}^\star F + \frac{1}{2f_{\rm e}^2}\, {}^\star(F \wedge F)\, F}{\sqrt{1 - \frac{1}{f_{\rm e}^2}\, {}^\star(F \wedge {}^\star F) - \frac{1}{4f_{\rm e}^4}\, [{}^\star(F \wedge F)]^2}}\,. \qquad \text{(E.2.6)}$$ BI

Because of the nonlinearity, the field of a point charge, for example, turns out to be finite at $r = 0$, in contrast to the well known $1/r^2$ singularity of the Coulomb field in the Maxwell-Lorentz electrodynamics, see Fig. E.2.1. The dimensionful parameter $f_{\rm e} = E_{\rm e}/c$ is defined by the so-called maximal field strength achieved in the Coulomb configuration of an electron: $E_{\rm e} = e/4\pi\varepsilon_0 r_0^2$, where $r_0 = \xi r_e$ with the classical electron radius $r_e = \alpha_{\rm f}\hbar/mc$ and a numerical constant $\xi \approx 1$. Explicitly, we have $E_{\rm e} \approx 1.8 \times 10^{20}$V/m and $f_{\rm e} = B_{\rm k}/\alpha_{\rm f} = \frac{m^2c^2}{\alpha_{\rm f} e\hbar} \approx 6.4 \times 10^{11}$ T.

In the quantum string theory, the Born-Infeld spacetime relation arises as an effective model with $f_{\rm e} = 1/2\pi\alpha'$ (where α' is the inverse string tension constant).

The spacetime relation (E.2.6) leads to a non-linear equation for the dynamical evolution of the field strength F. As a consequence, the characteristic surface, the light cone, depends on the field strength, and the superposition principle for the electromagnetic field doesn't hold any longer.

[3] See also Gibbons and Rasheed [6].

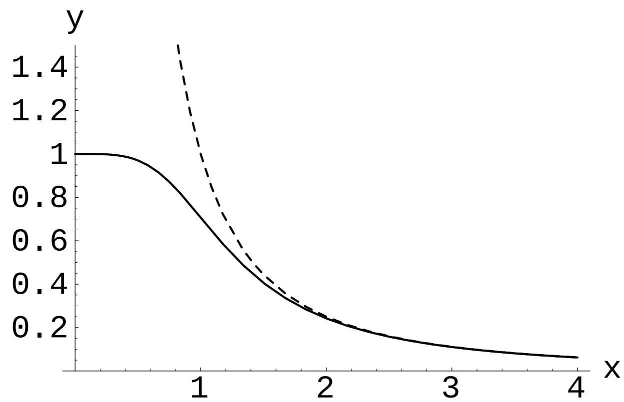

Figure E.2.1: Spherically symmetric electric field of a point charge in the Born-Infeld (solid line) and in Maxwell-Lorentz theory (dashed line). On the axes we have dimensionless variables $x = r/r_0$ and $y = E/E_e$.

E.2.5 ⊗Plebański

Both, Eqs.(E.2.6) and (E.2.5) are special cases of Plebański's more general non-linear electrodynamics [17]. Let the quadratic invariants of the electromagnetic field strength be denoted by

$$I_1 := \frac{1}{2}{}^\star(F \wedge {}^\star F) = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \quad \text{and} \quad I_2 := \frac{1}{2}{}^\star(F \wedge F) = \vec{E} \cdot \vec{B}\,, \tag{E.2.7}$$

Inv

where I_1 is an even and I_2 is an odd scalar (the Hodge operator is odd). Then Plebański postulated a non-linear electrodynamics with the spacetime relation[4]

$$H = U(I_1, I_2)\,{}^\star F + V(I_1, I_2)\,F\,, \tag{E.2.8}$$

non-l

where U and V are functions of the two invariants. Note that in the Born-Infeld case U and V depend on both invariants whereas in the Heisenberg-Euler case we have $U_{\mathrm{HE}}(I_1)$ and $V_{\mathrm{HE}}(I_2)$. Nevertheless, in both cases U is required as well as V. And in both

---

[4] Strictly, Plebański assumed a Lagrangian which yields the Maxwell equations together with the *structural relations* $F = u(I_1, I_2)\,{}^\star H + v(I_1, I_2)\,H$. The latter law, apart from singular cases, is equivalent to (E.2.8).

cases, see (E.2.6) and (E.2.5), U is an even function and V and odd one such as to preserve parity invariance. If one chooses $V(I_1, I_2)$ to be an even function, then parity violating terms would emerge, a case which is not visible in experiment.

E.3

Electrodynamics in matter, constitutive law

E.3.1 Splitting of the current: Sixth axiom

In this chapter we will present a consistent *microscopic* approach to the electrodynamics in continuous media[1]. Besides the field strength F, the excitation H is a microscopic field in its own right, as we have shown in our axiomatic discussion in Part B. The total current density is the sum of the two contributions originating "from the inside" (bound charge) and "from the outside" (free charge) of the medium:

$$J = J^{\mathrm{mat}} + J^{\mathrm{ext}} \qquad \text{(sixth axiom a)}. \tag{E.3.1}$$ total

Accordingly, the bound electric current in matter is denoted by *mat* and the external current by *ext*. The same notational scheme will also be applied to the excitation H; we will have H^{mat} and H^{ext}.

[1] In a great number of the texts on electrodynamics the electric and magnetic properties of media are described following the *macroscopic averaging* scheme of Lorentz [14]. However, this formalism has a number of serious limitations, see the relevant criticism of Hirst [11], e.g.. An appropriate modern presentation of the microscopic approach to this subject has been given in the textbook of Kovetz [13].

Bound charges and bound currents are inherent characteristics of matter determined by the medium itself. They only emerge *inside* the medium. In contrast, external charges and external currents in general appear outside and inside matter. They can be prepared for a specific purpose by a suitable experimental arrangement. we can, for instance, prepare a beam of charged particles, described by $J^{\rm ext}$, and can scatter them at the medium, or we could study the reaction of a medium in response to a prescribed configuration of charges and currents, $J^{\rm ext}$.

We assume that the charge bound by matter fulfills the usual charge conservation law separately:

$$d\,J^{\rm mat} = 0 \qquad \text{(sixth axiom b)}. \tag{E.3.2}$$ Axiom6

We will call this relation as the *6th axiom* which specifies the properties of the classical material medium. In view of the first axiom (B.1.19), this assumption means that there is no physical exchange (or conversion) between the bound and the external charges. The 6th axiom certainly does not exhaust all possible types of material media, but it is valid for a wide enough class of continua.

Mathematically, the 6th axiom (E.3.2) has the same form as the 1st axiom. As a consequence, we can repeat the arguments of Sec. B.1.3 and will find the corresponding excitation $H^{\rm mat}$ as a "potential" for the bound current:

$$J^{\rm mat} = d\,H^{\rm mat}\,. \tag{E.3.3}$$ curexactM

The $(1+3)$-decomposition, following the pattern of (B.1.36), yields

$$H^{\rm mat} = {}^{\perp}H^{\rm mat} + \underline{H}^{\rm mat} = d\sigma \wedge \mathcal{H}^{\rm mat} + \mathcal{D}^{\rm mat}. \tag{E.3.4}$$ decomexiM

The conventional names for these newly introduced excitations are *polarization* 2-form P and *magnetization* 1-form M , i.e.,

$$\mathcal{D}^{\rm mat} \equiv -\,P\,, \qquad \mathcal{H}^{\rm mat} \equiv M\,. \tag{E.3.5}$$ PM

The minus sign is conventional. Thus, in analogy to the inhomogeneous Maxwell equations (B.1.40)-(B.1.41), we find

$$\underline{d}\,M + \dot{P} = j^{\mathrm{mat}}, \qquad -\,\underline{d}\,P = \rho^{\mathrm{mat}}\,. \tag{E.3.6}$$ dP

The identifications (E.3.5) are only true up to an exact form. However, if we require $\mathcal{D}^{\mathrm{mat}} = 0$ for $E = 0$ and $\mathcal{H}^{\mathrm{mat}} = 0$ for $B = 0$, as we will do in (E.3.14), uniqueness is guaranteed.

E.3.2 Maxwell's equations in matter

The Maxwell equations (B.4.8)

$$dH = J \quad \text{or} \quad \left\{ \begin{array}{l} \underline{d}\,\mathcal{H} - \dot{\mathcal{D}} = j \\ \underline{d}\,\mathcal{D} = \rho \end{array} \right. \tag{E.3.7}$$ evol1a

are *linear* partial differential equations of first order. Therefore it is useful to define the external excitation

$$\mathbb{H} := H - H^{\mathrm{mat}} \quad \text{or} \quad \left\{ \begin{array}{l} \mathfrak{D} := \mathcal{D} - \mathcal{D}^{\mathrm{mat}} = \mathcal{D} + P \\ \mathfrak{H} := \mathcal{H} - \mathcal{H}^{\mathrm{mat}} = \mathcal{H} - M \end{array} \right\}. \tag{E.3.8}$$ DHe

The excitation $\mathbb{H} = (\mathfrak{D}, \mathfrak{H})$ can be understood as an auxiliary quantity. We differentiate (E.3.8) and eliminate dH and dH^{mat} by (E.3.7) and (E.3.3), respectively. Then using (E.3.1), the *inhomogeneous* Maxwell equation for matter finally reads

$$d\mathbb{H} = J^{\mathrm{ext}}, \tag{E.3.9}$$ MaxMat

or, in $(1+3)$-decomposed form,

$$\underline{d}\,\mathfrak{D} = \rho^{\mathrm{ext}}\,, \tag{E.3.10}$$ dDe

$$\underline{d}\,\mathfrak{H} - \dot{\mathfrak{D}} = j^{\mathrm{ext}}. \tag{E.3.11}$$ dHe

From (E.3.8) and the universal spacetime relation (E.1.19) we obtain

$$\mathfrak{D} = \varepsilon_{\mathrm{g}}\varepsilon_0\, \underline{*}E + P[E,B]\,, \tag{E.3.12}$$

$$\mathfrak{H} = \frac{1}{\mu_{\mathrm{g}}\mu_0}\, \underline{*}B - M[B,E]\,. \tag{E.3.13}$$

The polarization $P[E,B]$ is a functional of the electromagnetic field strengths E and B. In general, it can depend also on the temperature T and possibly on other thermodynamic variables specifying the material continuum under consideration; similar remarks apply to the magnetization $M[B,E]$. The system (E.3.10)-(E.3.11) looks similar to the Maxwell equations (E.3.7). However, the former equations refer only to the external fields and sources. The *homogeneous* Maxwell equation remains valid in its original form.

E.3.3 Linear constitutive law

"It should be needless to remark that while from the mathematical standpoint a constitutive equation is a postulate or a definition, the first guide is physical experience, perhaps fortified by experimental data." C. Truesdell and R.A. Toupin (1960)

In the simplest case of *isotropic homogeneous media at rest* with nontrivial polarizational/magnetizational properties, we have the linear constitutive laws

$$P = \varepsilon_{\rm g}\varepsilon_0\,\chi_{\rm E}\,{}^{\underline{*}}E\,, \qquad M = \frac{1}{\mu_{\rm g}\mu_0}\,\chi_{\rm B}\,{}^{\underline{*}}B\,, \qquad \text{(E.3.14)}$$

susc

with the electric and magnetic susceptibilities $(\chi_{\rm E},\chi_{\rm B})$. If we introduce the material constants

$$\varepsilon := 1+\chi_{\rm E}\,, \qquad \mu := \frac{1}{1-\chi_{\rm B}}\,, \qquad \text{(E.3.15)}$$

epsmu

one can rewrite the constitutive laws (E.3.14) as

$$\mathfrak{D} = \varepsilon\varepsilon_{\rm g}\,\varepsilon_0\,{}^{\underline{*}}E \qquad \text{and} \qquad B = \mu\mu_{\rm g}\,\mu_0\,{}^{\underline{*}}\mathfrak{H}\,. \qquad \text{(E.3.16)}$$

perm

In curved spacetime, the quantities $(\varepsilon\varepsilon_{\rm g})$ and $(\mu\mu_{\rm g})$, in general, are the functions of coordinates, but in flat Minkowski spacetime they are usually constant. However, $\varepsilon \neq \mu$, contrary to the effective gravitational permeabilities (E.1.20). In general case, their values are determined by the electric and magnetic polarizability of a material medium.

A medium characterized by (E.3.16) is called a *simple*. For a conductive medium, one usually adds one more constitutive relation, namely, Ohm's law:

$$j = \sigma\,\underline{{}^\star E}. \tag{E.3.17}$$

ohm

Here σ is the conductivity of the simple medium.

An alternative way of writing the constitutive law (E.3.16) is by using the foliation projectors explicitly:

$$\underline{\mathbb{H}} = \varepsilon\,\lambda\,\underline{({}^\star F)}, \qquad \mu\,{}^\perp\mathbb{H} = \lambda\,{}^\perp({}^\star F). \tag{E.3.18}$$

const4

This form is particularly convenient for the discussion of the transition to vacuum. Then $\varepsilon = \mu = 1$ and hence (E.3.18) immediately reduces to the universal spacetime relation (D.5.7).

For *anisotropic* media the constitutive laws (E.3.16)-(E.3.17) are further generalized by replacing ε, μ, σ by the linear operators $\boldsymbol{\varepsilon}, \boldsymbol{\mu}, \boldsymbol{\sigma}$ acting on the spaces of transversal 2- and 1-forms. The easiest way to formulate these constitutive laws explicitly is to use the vector components of the electromagnetic fields and excitations which were earlier introduced in (D.1.37)-(D.1.38). In this description, the linear operators are just 3×3 matrices $\boldsymbol{\varepsilon} = \varepsilon^{ab}$ and $\boldsymbol{\mu} = \mu_{ab}$. One can write explicitly

$$\begin{pmatrix} \mathfrak{H}_a \\ \mathfrak{D}^a \end{pmatrix} = \lambda \begin{pmatrix} 0 & \frac{N}{\sqrt{g}}\,\mu_{ab} \\ -\frac{\sqrt{g}}{N}\,\varepsilon^{ab} & 0 \end{pmatrix} \begin{pmatrix} -E_b \\ B^b \end{pmatrix}. \tag{E.3.19}$$

lingen

In general, the matrices ε^{ab} and μ_{ab} depend on the spacetime coordinates. The contribution of the gravitational field is included in the metric-dependent factors N and $\sqrt{g} = \sqrt{{}^{(3)}g}$ [In Minkowski spacetime, $N = c, \sqrt{g} = 1$].

E.3.4 Energy-momentum currents in matter

In a medium, the total electric current J is the sum (E.3.1) of the external or free charge $J^{\rm ext}$ and the material or bound charge contributions $J^{\rm mat}$. Thus, one should carefully distinguish two different physical situations: (i) when we are inspecting how the electric and magnetic fields act on the *external or free charges and currents* (which are used in actual observations in media, e.g.) and (ii) when we study the influence of the electromagnetic field on the *material or bound charges and currents* (i.e., on dielectric and magnetic bodies).

Correspondingly, we have to consider the Lorentz force density $f_\alpha^{\rm ext} = (e_\alpha \rfloor F)\wedge J^{\rm ext}$ which affects the external (free) current $J^{\rm ext}$ and the Lorentz force density $f_\alpha^{\rm mat} = (e_\alpha \rfloor F) \wedge J^{\rm mat}$ which affects the material (bound) charges. We can study these two situations separately because, as we assumed in Sec. E.3.1, there is no physical mixing between the free and the bound currents, in particular, they are conserved separately.

For the first case, by using the inhomogeneous Maxwell equations in matter, (E.3.9) or (E.3.10)-(E.3.11), and repeating the derivations of Sec. B.5.3, we obtain

$$f_\alpha^{\rm ext} = (e_\alpha \rfloor F) \wedge J^{\rm ext} = d\,{}^{\rm f}\Sigma_\alpha + {}^{\rm f}X_\alpha\,. \tag{E.3.20}$$

fSXe

Here the *"free-charge" energy-momentum* 3-form of the electromagnetic field and the supplementary term are, respectively, given by

$$\begin{aligned} {}^{\rm f}\Sigma_\alpha &:= \frac{1}{2}\left[F\wedge(e_\alpha \rfloor \mathbb{H}) - \mathbb{H}\wedge(e_\alpha \rfloor F)\right] , &\text{(E.3.21)}\\ {}^{\rm f}X_\alpha &:= -\frac{1}{2}\left(F\wedge \mathcal{L}_{e_\alpha}\mathbb{H} - \mathbb{H}\wedge \mathcal{L}_{e_\alpha}F\right) . &\text{(E.3.22)} \end{aligned}$$

sigmaM
XaM

This energy-momentum describes the action of the electromagnetic field on the free charges (hence the notation where superscript "f" stands for "free").

At first, let us analyze the supplementary term. In $(1+3)$-decomposed form, we have $F = E\wedge d\sigma + B$, and $\mathbb{H} = -\mathfrak{H}\wedge d\sigma + \mathfrak{D}$. The Lie derivatives of these 2-forms can be easily computed,

$$\begin{aligned} \mathcal{L}_{e_0}F &= \dot{E}\wedge d\sigma + \dot{B},\\ \mathcal{L}_{e_a}F &= (\underline{\mathcal{L}}_{e_a}E)\wedge d\sigma + \underline{\mathcal{L}}_{e_a}B, \end{aligned} \tag{E.3.23}$$

LieF

and similarly,

$$\begin{aligned} \mathcal{L}_{e_0} I\!H &= -\,\dot{\mathfrak{H}} \wedge d\sigma + \dot{\mathfrak{D}}, \\ \mathcal{L}_{e_a} I\!H &= -\,(\underline{\mathcal{L}}_{e_a} \mathfrak{H}) \wedge d\sigma + \underline{\mathcal{L}}_{e_a} \mathfrak{D}. \end{aligned} \qquad \text{(E.3.24)} \quad \texttt{LieHe}$$

Here $\underline{\mathcal{L}}_{e_a} := \underline{d} e_a \rfloor + e_a \rfloor \underline{d}$ is the purely spatial Lie derivative. Substituting (E.3.23)-(E.3.24) into (E.3.22), we find

$$\begin{aligned} {}^{\rm f}\!X_0 = \frac{1}{2}\, d\sigma \wedge \Big[&\mathfrak{H} \wedge \dot{B} - \dot{\mathfrak{H}} \wedge B \\ &+ E \wedge \dot{\mathfrak{D}} - \dot{E} \wedge \mathfrak{D} \Big], \end{aligned} \qquad \text{(E.3.25)} \quad \texttt{X0mat}$$

$$\begin{aligned} {}^{\rm f}\!X_a = \frac{1}{2}\, d\sigma \wedge \Big[&\mathfrak{H} \wedge \underline{\mathcal{L}}_{e_a} B - \underline{\mathcal{L}}_{e_a} \mathfrak{H} \wedge B \\ &+ E \wedge \underline{\mathcal{L}}_{e_a} \mathfrak{D} - \underline{\mathcal{L}}_{e_a} E \wedge \mathfrak{D} \Big]. \end{aligned} \qquad \text{(E.3.26)} \quad \texttt{Xamat}$$

In Minkowski spacetime for matter with the general linear constitutive law (E.3.19), we find

$${}^{\rm f}\!X_\alpha = \frac{1}{2}\,\mathrm{Vol} \left[\varepsilon_0\, (\partial_\alpha \varepsilon^{ab})\, E_a E_b - \frac{1}{\mu_0}\, (\partial_\alpha \mu_{ab})\, B^a B^b \right], \qquad \text{(E.3.27)} \quad \texttt{Xepmu}$$

where Vol is the 4-form of the spacetime volume. Thus ${}^{\rm f}\!X_\alpha$ vanishes for homogeneous media with constant electric and magnetic permeabilities.

The structure of the free-charge energy-momentum is revealed via the standard $(1+3)$-decomposition:

$${}^{\rm f}\Sigma_{\hat 0} = {}^{\rm f}u - d\sigma \wedge {}^{\rm f}s\,, \qquad \text{(E.3.28)} \quad \texttt{sig0ext}$$

$${}^{\rm f}\Sigma_a = {}^{\rm f}p_a - d\sigma \wedge {}^{\rm f}S_a\,. \qquad \text{(E.3.29)} \quad \texttt{sigaext}$$

Here, in complete analogy with (B.5.50)-(B.5.51) and (B.5.52)-(B.5.55), we introduced the *energy* density 3-form

$${}^{\rm f}u := \frac{1}{2}\,(E \wedge \mathfrak{D} + B \wedge \mathfrak{H})\,, \qquad \text{(E.3.30)} \quad \texttt{enerExt}$$

the *energy flux* density (or Poynting) 2-form

$${}^{\rm f}s := E \wedge \mathfrak{H}\,, \qquad \text{(E.3.31)} \quad \texttt{poyntExt}$$

the *momentum* density 3-form

$$ {}^{\rm f}p_a := B \wedge (e_a \rfloor \mathfrak{D})\,, \tag{E.3.32}$$ momExt

and the *stress* (or momentum flux density) 2-form of the electromagnetic field

$$\begin{aligned} {}^{\rm f}S_a :=& \frac{1}{2}\left[(e_a \rfloor E) \wedge \mathfrak{D} - (e_a \rfloor \mathfrak{D}) \wedge E \right.\\ &\left. + (e_a \rfloor \mathfrak{H}) \wedge B - (e_a \rfloor B) \wedge \mathfrak{H}\right]. \end{aligned} \tag{E.3.33}$$ stressExt

In absence of free charges and currents, we have the balance equations for the electromagnetic field energy and momentum $d\,{}^{\rm f}\Sigma_\alpha + {}^{\rm f}X_\alpha = 0$. In the (1 + 3)-decomposed form this reads, analogously to (B.5.62)-(B.5.63):

$$ {}^{\rm f}\dot{u} + \underline{d}\,{}^{\rm f}s + ({}^{\rm f}X_{\hat{0}})_\perp = 0, \tag{E.3.34}$$ k0ext

$$ {}^{\rm f}\dot{p}_a + \underline{d}\,{}^{\rm f}S_a + ({}^{\rm f}X_a)_\perp = 0. \tag{E.3.35}$$ kaext

The (Minkowski) energy-momentum (E.3.21) is associated with free charges and has no relation to the forces acting on dielectric and/or magnetic bodies and media. It is, however, indispensable for analyzing the wave phenomenae in matter.

It seems worthwhile to mention that there is long controversy concerning the lack of symmetry of the Minkowski[2] energy-momentum tensor. Let us study this question in Minkowski spacetime for the case of an *isotropic medium* at rest in the laboratory frame. Using the definitions (E.1.29) and (E.3.21), we find for the "off-diagonal" components:

$$ {}^{\rm f}T_0{}^a = -\,\eta^{abc}\,\mathfrak{H}_b E_c, \quad {}^{\rm f}T_a{}^0 = \frac{\varepsilon\mu}{c^2}\,\eta_{abc}\,\mathfrak{H}^b E^c. \tag{E.3.36}$$

If the upper index is now lowered with the help of the spacetime metric, ${}^{\rm f}T_i{}^k\, g_{kj}$, then we find indeed that

$$ {}^{\rm f}T_i{}^k\, g_{kj} \neq {}^{\rm f}T_j{}^k\, g_{ki}, \tag{E.3.37}$$

[2]Abraham proposed a symmetric energy-momentum tensor which turned out to be *obsolete*, in contrast to repeated claims in the literature (see [1], e.g.) to the opposite, see below.

because ${}^{\rm f}T_0{}^b\, g_{ba} = \eta_{abc}\, \mathfrak{H}^b E^c$, whereas ${}^{\rm f}T_a{}^0\, g_{00} = \varepsilon\mu\, \eta_{abc}\, \mathfrak{H}^b E^c$. The extra factor is just the square of the refractive index $n^2 = \varepsilon\mu$ of matter. However it is remarkable that the Minkowski energy-momentum tensor is symmetric, provided we use the *optical metric* for the lowering of the upper index:

$$ {}^{\rm f}T_i{}^k\, g^{\rm opt}_{kj} = {}^{\rm f}T_j{}^k\, g^{\rm opt}_{ki}. \qquad \text{(E.3.38)} $$

maxsymopt

One can easily verify this by means of (E.4.35). The *generalized symmetry* (E.3.38) holds true also for an arbitrarily moving medium. Then the optical metric is described by (E.4.33). This fact highlights the fundamental position which the optical metric occupies in the Maxwell-Lorentz theory. Thus it is not by chance that the Minkowski energy-momentum turns out to be most useful for the discussion of the optical phenomenae in material media.

Let us now turn our attention back to the forces $f^{\rm mat}_\alpha = (e_\alpha \rfloor F)\wedge J^{\rm mat}$ acting on the bound charges. In complete analogy with the derivation of (E.3.20)-(E.3.22), we find

$$ f^{\rm mat}_\alpha = (e_\alpha \rfloor F) \wedge J^{\rm mat} = d\, {}^{\rm b}\Sigma_\alpha + {}^{\rm b}X_\alpha\,. \qquad \text{(E.3.39)} $$

fSXb

Here we introduce a new *material energy-momentum* 3-form of the electromagnetic field and the corresponding supplementary term:

$$ {}^{\rm b}\Sigma_\alpha := \frac{1}{2}\left[F \wedge (e_\alpha \rfloor H^{\rm mat}) - H^{\rm mat} \wedge (e_\alpha \rfloor F)\right]\,, \qquad \text{(E.3.40)} $$

sigma-b

$$ {}^{\rm b}X_\alpha := -\frac{1}{2}\left(F \wedge \mathcal{L}_{e_\alpha} H^{\rm mat} - H^{\rm mat} \wedge \mathcal{L}_{e_\alpha} F\right)\,. \qquad \text{(E.3.41)} $$

Xa-b

The material energy-momentum describes the action of the electromagnetic field on the bound charges (hence the notation where superscript "b" stands for "bound"). In (1+3)-decomposed form, we have $H^{\rm mat} = -(M \wedge d\sigma + P)$ and, as usual, $F = E \wedge d\sigma + B$. Thus the energy-momentum (E.3.40) is ultimately expressed in terms of the polarization P and magnetization M forms (E.3.5). When there are no free charges and currents, we should use (E.3.40), and not the energy-momentum (E.3.21), for the computation of the forces acting on the dielectric and magnetic matter.

The (1+3)-decomposition yields a similar structure as (E.3.28)-(E.3.29):

$$ {}^{\rm b}\Sigma_{\hat 0} = {}^{\rm b}u - d\sigma \wedge {}^{\rm b}s \,, \qquad \text{(E.3.42)} \quad \texttt{sig0mat} $$

$$ {}^{\rm b}\Sigma_{a} = {}^{\rm b}p_a - d\sigma \wedge {}^{\rm b}S_a \,. \qquad \text{(E.3.43)} \quad \texttt{sigamat} $$

Here, in complete analogy with (B.5.50)-(B.5.51) and (B.5.52)-(B.5.55), we introduced the *bound-charge energy* density 3-form

$$ {}^{\rm b}u := \frac{1}{2}\left(B\wedge M - E\wedge P\right) , \qquad \text{(E.3.44)} \quad \texttt{enerMat} $$

the *bound-charge energy flux* density 2-form

$$ {}^{\rm b}s := E\wedge M \,, \qquad \text{(E.3.45)} \quad \texttt{poyntMat} $$

the *bound-charge momentum* density 3-form

$$ {}^{\rm b}p_a := -\,B\wedge (e_a \rfloor P)\,, \qquad \text{(E.3.46)} \quad \texttt{momMat} $$

and the *bound-charge stress* (or momentum flux density) 2-form of the electromagnetic field

$$ \begin{aligned} {}^{\rm b}S_a := \frac{1}{2}\Big[&(e_a \rfloor P)\wedge E - (e_a \rfloor E)\wedge P \\ &+ (e_a \rfloor M)\wedge B - (e_a \rfloor B)\wedge M\Big] . \end{aligned} \qquad \text{(E.3.47)} \quad \texttt{stressMat} $$

The (1+3)-decomposed balance equations for the bound-charge energy-momentum are analogous to (B.5.62)-(B.5.63) and (E.3.34)-(E.3.35):

$$ {}^{\rm b}k_0 = {}^{\rm b}\dot u + \underline{d}\,{}^{\rm b}s + ({}^{\rm b}X_{\hat 0})_\perp , \qquad \text{(E.3.48)} \quad \texttt{k0mat} $$

$$ {}^{\rm b}k_a = {}^{\rm b}\dot p_a + \underline{d}\,{}^{\rm b}S_a + ({}^{\rm b}X_a)_\perp . \qquad \text{(E.3.49)} \quad \texttt{kamat} $$

The integral of the 3-form of the force density (E.3.49) over the 3-dimensional domain $\Omega^{\rm mat}$ occupied by a material body or a medium gives the total 3-force acting on the latter:

$$ K_a = \int\limits_{\Omega^{\rm mat}} {}^{\rm b}k_a . \qquad \text{(E.3.50)} \quad \texttt{totforce} $$

E.3.5 ⊗Experiment of Walker & Walker

Let us consider an explicit example which shows how the energy-momentum current (E.3.40) works. For concreteness, we will analyze the experiment of Walker & Walker who measured the force acting on a dielectric disc placed in a vertical magnetic field (i.e., between the poles of the electromagnet) as shown on Fig. E.3.1. The time-dependent magnetic field was synchronized with the alternating voltage applied to the inner and outer cylindrical surfaces of the disc at radius ρ_1 and ρ_2, respectively, thus creating the electric field along the radial direction. The experiment revealed the torque along the vertical z-axis. We will derive this torque by using the bound-charge energy-momentum current.

We have the Minkowski spacetime geometry. In cylindrical coordinates (ρ, φ, z), the torque density along the z-axis evidently is given by product $\rho\, {}^{\mathrm{b}}\mathfrak{k}_{\varphi}$. Hence the total torque is the integral

$$N^z = \int\limits_{\mathrm{Disc}} \rho\, {}^{\mathrm{b}}\mathfrak{k}_{\varphi}. \tag{E.3.51}$$

torque

Assuming the harmonically oscillating electric and magnetic fields, we find for the excitations inside the disc

$$\mathfrak{D} = \lambda n\, \sin(\omega t)\, dz \wedge \left[a_1\, J_1\Big(\frac{\rho\omega n}{c}\Big)\, d\rho - \frac{a_2}{\rho}\, \cos\Big(\frac{z\omega n}{c}\Big)\, \rho\, d\varphi\right], \tag{E.3.52}$$

walD

$$\mathfrak{H} = \frac{\cos(\omega t)}{\mu_0}\left[a_1\, J_0\Big(\frac{\rho\omega n}{c}\Big)\, dz - \frac{a_2}{\rho}\, \sin\Big(\frac{z\omega n}{c}\Big)\, \rho\, d\varphi\right]. \tag{E.3.53}$$

walH

Here $\lambda = \sqrt{\varepsilon_0/\mu_0}$ and $n := \sqrt{\varepsilon}$ is the refractive index of the medium. The disc consists of nonmagnetic dielectric material with $\mu = 1$. The oscillation frequency is ω, the two constants a_1, a_2 determine the magnitude of electromagnetic fields, and

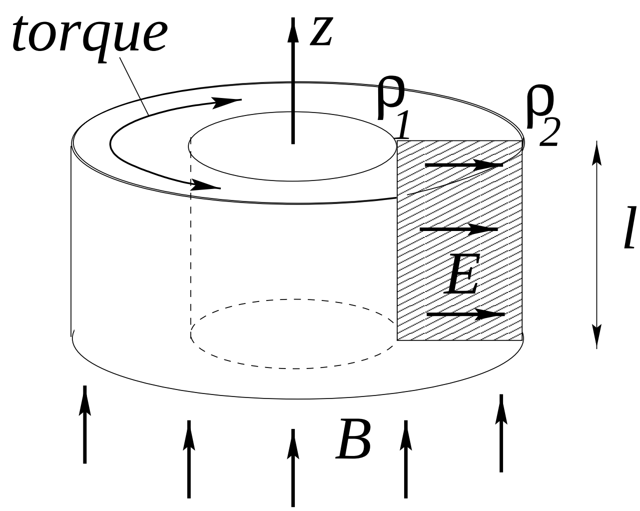

Figure E.3.1: Experiment of Walker & Walker

J_0, J_1 are Bessel functions. The field strength forms look similar:

$$E = \sin(\omega t)\, \frac{c}{n} \left[a_1\, J_1\!\left(\frac{\rho\omega n}{c}\right) \rho\, d\varphi + \frac{a_2}{\rho}\, \cos\!\left(\frac{z\omega n}{c}\right) d\rho \right], \tag{E.3.54}$$ walE

$$B = \cos(\omega t)\, d\rho \wedge \left[a_1\, J_0\!\left(\frac{\rho\omega n}{c}\right) \rho\, d\varphi + \frac{a_2}{\rho}\, \sin\!\left(\frac{z\omega n}{c}\right) dz \right]. \tag{E.3.55}$$ walB

One can verify by substitution that the electromagnetic field (E.3.52)-(E.3.55) represents an exact solution of the Maxwell equations plus the constitutive relations (E.3.16).

In the actual Walker & Walker experiment[3], the disc, made of barium titanate with $\varepsilon = 3340$, has $l \approx 2$ cm height and the internal and external radius $\rho_1 \approx 0.4$ cm and $\rho_2 \approx 2.6$ cm, respectively. The oscillation frequency is rather low, $\omega = 60$ Hz. Correspondingly, one can verify that everywhere in the disc we have

$$\frac{\rho\omega n}{c} \sim \frac{z\omega n}{c} \sim 10^{-7} \ll 1. \tag{E.3.56}$$

[3] See Walker and Walker [22]

Then the field strengths read approximately

$$E = \sin(\omega t)\left[\frac{1}{2}\,a_1\,\omega\,\rho^2\,d\varphi + \frac{ca_2}{n\rho}\,d\rho\right], \qquad \text{(E.3.57)}$$ walEa

$$B = \cos(\omega t)\left[a_1\,d\rho\wedge\rho d\varphi - \frac{ca_2}{n\rho}\,\omega z\,dz\wedge d\rho\right]. \qquad \text{(E.3.58)}$$ walBa

Using the constitutive relations (E.3.14), we find the polarization 2-form

$$P = \varepsilon_0\chi_{\rm E}\,\sin(\omega t)\left[\frac{1}{2}\,a_1\,\omega\,\rho\,dz\wedge d\rho + \frac{ca_2}{n}\,d\varphi\wedge dz\right]. \qquad \text{(E.3.59)}$$ walP

The 1-form of magnetization is vanishing, $M = 0$, since $\mu = 1$. For the computation of the torque around the vertical axis, we need only the azimuthal components of the momentum (E.3.46) and of the stress form (E.3.47). Note that $e_\varphi = \frac{1}{\rho}\partial_\varphi$. A simple calculation yields:

$$ {}^{\rm b}p_\varphi = -\,\varepsilon_0\chi_{\rm E}\,\sin(\omega t)\cos(\omega t)\frac{ca_1a_2}{n\rho}\,d\rho\wedge\rho d\varphi\wedge dz, \qquad \text{(E.3.60)}$$

$$\underline{d}\,{}^{\rm b}S_\varphi = -\,\varepsilon_0\chi_{\rm E}\,\sin^2(\omega t)\,\frac{c\omega a_1a_2}{n\rho}\,d\rho\wedge\rho d\varphi\wedge dz. \qquad \text{(E.3.61)}$$

Substituting this into (E.3.49) and subsequently computing the integral (E.3.51), we find the torque

$$N^z = -\,\varepsilon_0(\varepsilon-1)\,\pi\,l\,(\rho_2^2-\rho_1^2)\,a_1a_2\,\frac{c\omega}{n}\,\cos^2(\omega t). \qquad \text{(E.3.62)}$$ walT

Here l is the height of the disc, whereas ρ_1 and ρ_2 are its inner and outer radius, as shown on Fig. E.3.1. Formula (E.3.62) was proven experimentally by Walker & Walker.

It is rather curious that this fact was considered as an argument in favour of the so-called Abraham energy-momentum tensor[4]. A formal coincidence is taking place, indeed, in the following sense: Recall that our starting point for the deriving the bound-charge energy-momentum was the Lorentz force (E.3.39). Quite generally, for the 3-force density (E.3.39), we have from (E.3.3)-(E.3.6):

$$\begin{aligned} f_a^{\rm mat} &= d\sigma\wedge\left(-\,B\wedge e_a\rfloor j^{\rm mat} + E\wedge e_a\rfloor\rho^{\rm mat}\right)\\ &= -\,d\sigma\wedge\Big(B\wedge e_a\rfloor\dot P\\ &\qquad + B\wedge e_a\rfloor\underline{d}M + E\wedge e_a\rfloor\underline{d}P\Big). \end{aligned} \qquad \text{(E.3.63)}$$ forceLP

[4]See [22] and [1], e.g.

In the Walker & Walker experiment, we have $M = 0$. By differentiating (E.3.59), one can prove that $\underline{d}P = 0$. Thus the last line in (E.3.63) vanishes. Accordingly, the force density reduces to a term $\sim B \times \dot{P}$ which resembles the so-called Abraham force. However, our derivation was not based on the Abraham energy-momentum and moreover, the argument of the symmetry of the energy-momentum tensor is absolutely irrelevant. As one can see, our bound-charge energy-momentum is manifestly asymmetric since energy flux (E.3.45) is plainly zero whereas the momentum (E.3.46) is nonvanishing.

E.4

Electrodynamics of moving continua

E.4.1 Laboratory and material foliation

The electric and magnetic parts of current, excitation, and field strength are only determined with respect to a certain foliation of spacetime. In Sec. B.1.4 we assumed the existence of a foliation specified by a formal "time" parameter σ and a vector field n. We know of how to express all the physical and geometrical objects in terms of their projections into transversal and longitudinal parts by using the coordinate-free $(1+3)$-decomposition technique, see Sec. B.1.4. The original spacetime foliation will be called a *laboratory foliation*.

Moving *macroscopic* matter, by means of its own velocity, defines another $(1+3)$-splitting of spacetime which is different from the original foliation discussed above. Here we will describe this *material foliation* and its inter-relation with the laboratory foliation.

Let us denote a 3-dimensional matter-filled domain by V. Mathematically, one starts with a 3-dimensional arithmetic space R^3 equipped with the coordinates ξ^a, where $a = 1, 2, 3$, and considers a smooth mapping $x_{(0)} : R^3 \to V \in X_4$ into spacetime which defines a 3-dimensional domain (hypersurface) V representing the initial distribution of matter. The coordinates ξ^a (known as the *Lagrange* coordinates in continuum mechanics) serve as labels which denote the elements of the material medium.

Given the initial configuration V of matter, we parametrize the dynamics of the medium by the coordinate τ which is defined as the proper time measured along an element's world line from the original hypersurface V. The resulting local coordinates (τ, ξ^a) are usually called the normalized comoving coordinates. The motion of matter is thus described by the functions $x^i(\tau, \xi^a)$, and we subsequently define the (mean) velocity 4-vector field by

$$u := \partial_\tau = \left(\frac{dx^i}{d\tau}\right)_{\xi^a = const} \partial_i \,. \tag{E.4.1}$$

udef

By construction, this vector field is timelike and it is normalized according to

$$\mathbf{g}(u, u) = c^2. \tag{E.4.2}$$

uu=1

Evidently, a family of observers comoving with matter is characterized by the same timelike congruence $x^i(\tau, \xi^a)$. They are making physical (in particular, electrodynamical) measurements in their local reference frames drifting with the material motion. By the *hypothesis of locality* it is assumed that the instruments in the comoving frame are not affected in an appreciable way by the local acceleration they experience. They measure the same as if they were in a suitable comoving instantaneous *inertial* frame.

After these preliminaries, we are ready to find the relation between the two foliations. Technically, the crucial point is to express the *laboratory* coframe $(d\sigma, \underline{dx}^a)$ in terms of the coframe adapted to the *material* foliation. Recall that according to our conventions formulated in Sec. B.1.4, $\underline{dx}^a = dx^a - n^a\, d\sigma$ is the transversal projection of the spatial coframe.

The motion of a medium uniquely determines the (1 + 3)-decomposition of spacetime through a *material foliation* which is obtained by replacing $n, d\sigma$ by $u, d\tau$. Note that the proper time differential is $d\tau = c^{-2}\, u_i\, dx^i$. Thus evidently

$$u \rfloor d\tau = 1. \tag{E.4.3}$$

norm1

We consider an arbitrary motion of the matter. The velocity field u is arbitrary, and one does *not* assume that the laboratory and moving reference systems are related by a Lorentz transformation.

The technique of the $(1+3)$-splitting is similar to that described in Sec. B.1.4 for the laboratory foliation. Namely, following the pattern of (B.1.22) and (B.1.23), one defines the decompositions with respect to the material foliation: For any p-form Ψ we denote the part longitudinal to the velocity vector u by

$${}^{\dashv}\Psi := d\tau \wedge \Psi_{\vdash}, \qquad \Psi_{\vdash} := u \rfloor \Psi, \tag{E.4.4}$$

longiM

and the part transversal to the velocity u by

$$\underset{\smile}{\Psi} := u \rfloor (d\tau \wedge \Psi) = (1 - {}^{\dashv})\Psi, \qquad u \rfloor \underset{\smile}{\Psi} \equiv 0. \tag{E.4.5}$$

transM

Please note that the projectors are denoted now differently (${}^{\dashv}$ and $\underset{\smile}{\ }$) in order to distinguish them from the corresponding projectors (${}^{\perp}$ and $\underline{\ }$) of the laboratory foliation.

With the spacetime metric introduced on the X_4 by means of the Maxwell–Lorentz spacetime relation, we assume that the laboratory foliation is consistent with the metric structure in the sense outlined in Sec. E.1.3. In particular, taking into account (E.1.13) and (E.1.14), one finds the line element with respect to the laboratory foliation coframe:

$$ds^2 = N^2\, d\sigma^2 + g_{ab}\, \underline{dx}^a\, \underline{dx}^b = N^2\, d\sigma^2 - {}^{(3)}g_{ab}\, \underline{dx}^a\, \underline{dx}^b. \tag{E.4.6}$$

metF

Now it is straightforward to find the relation between the two foliations. Technically, by using (E.4.4) and (E.4.5), one just needs to $(1+3)$-decompose the basis 1-forms of the laboratory coframe $(d\sigma, \underline{dx}^a)$ with respect to the material foliation. Taking into account that, in the local coordinates, $n = \partial_\sigma + n^a \partial_a$ and

Table E.4.1: *Two foliations*

	laboratory frame	**material frame**
vector field	n	u
"time"	σ	τ
longitudinal	${}^{\perp}\Psi$	${}^{\dashv}\Psi$
transversal	$\underline{\Psi}$	$\underset{\sim}{\Psi}$
time coframe	$d\sigma$	$d\tau$
$3D$ coframe	$\underline{dx}^a$	$\underset{\sim}{dx}^a$
$4D$ spacetime interval	$ds^2 = N^2 d\sigma^2$ $-{}^{(3)}g_{ab}\underline{dx}^a\underline{dx}^b$	$ds^2 = c^2 d\tau^2$ $-\left({}^{(3)}g_{ab} - \frac{1}{c^2}\,v_a v_b\right)\underset{\sim}{dx}^a\underset{\sim}{dx}^b$

similarly $u = u^{(\sigma)}\partial_\sigma + u^a\partial_a$, the result, in a convenient matrix form, reads:

$$\begin{pmatrix} d\sigma \\ \underline{dx}^a \end{pmatrix} = \left(\begin{array}{c|c} \gamma c/N & v_b/(cN) \\ \hline \gamma v^a & \delta^a_b \end{array}\right) \begin{pmatrix} d\tau \\ \underset{\sim}{dx}^b \end{pmatrix}. \tag{E.4.7}$$ dsigmaM

Here we introduced for the *relative velocity* 3–vector the notation

$$v^a := \frac{c}{N}\left(\frac{u^a}{u^{(\sigma)}} - n^a\right), \tag{E.4.8}$$ relvel

furthermore

$$\gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{E.4.9}$$ gamM

Observe that (E.4.7) is *not* a Lorentz transformation since it relates the two frames which are both *non*inertial in general.

As usual, the spatial indices are raised and lowered by the 3–space metric of (E.1.15), ${}^{(3)}g_{ab} := -g_{ab}$. In particular, we explicitly have $v_a = {}^{(3)}g_{ab}\,v^b$ and $v^2 := v_a v^a$. By means of the normalization (E.4.2), we can express the zeroth (time) component of the velocity as $u^{(\sigma)} = \gamma(c/N)$. Hence the explicit form of the matter 4-velocity reads:

$$u = u^{(\sigma)}\,\partial_\sigma + u^a\,\partial_a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{c}{N}\,e_{\widehat{0}} + v^a\,e_a\right). \qquad \text{(E.4.10)}$$ Umat

Here $(e_{\widehat{0}}, e_a)$ is the frame dual to the adapted laboratory coframe $(d\sigma, \underline{dx}^a)$, i.e., $e_{\widehat{0}} = n$, $e_a = \partial_a$.

When the relative 3–velocity is zero $v^a = 0$, the material and the laboratory foliations coincide because the corresponding foliation 1-forms turn out to be proportional $u = (c/N)\,n$.

Substituting (E.4.7) into (E.4.6), we find for the line element in terms of the new variables

$$ds^2 = c^2\,d\tau^2 - \widehat{g}_{ab}\,\underset{\sim}{dx}^a\,\underset{\sim}{dx}^b, \quad \text{where} \quad \widehat{g}_{ab} = {}^{(3)}g_{ab} - \frac{1}{c^2}\,v_a v_b. \qquad \text{(E.4.11)}$$ metM

Comparing this with (E.4.6), we clearly see that the transition from a laboratory reference frame to a moving material reference frame changes the form of the line element from (E.4.6) to (E.4.11). Consequently, this transition corresponds to a linear homogeneous transformation that is anholonomic in general. It is not a Lorentz transformation which, by definition, preserves the form of the metric coefficients.

The metric $\widehat{g}_{ab}$ of the material foliation has the inverse

$$\widehat{g}^{ab} = {}^{(3)}g^{ab} + \frac{\gamma^2}{c^2}\,v^a v^b. \qquad \text{(E.4.12)}$$ invghat

For its determinant one finds $(\det \widehat{g}_{ab}) = (\det g_{ab})\,\gamma^{-2}$.

E.4.2 Electromagnetic field in laboratory and material frames

Let us consider the case of a simple medium with homogeneous and isotropic electric and magnetic properties. The constitutive law (E.3.18) for such a *medium at rest with respect to the laboratory frame* has to be understood as a result of a laboratory foliation. A moving medium is naturally at rest with respect to its own material foliation. Consequently, the constitutive law for such a simple medium reads

$$\underset{\sim}{I\!H} = \varepsilon\,\lambda\,(\underset{\sim}{{}^\star F}), \qquad \mu^{-1} I\!H = \lambda^{-1}({}^\star F). \tag{E.4.13}$$ const5a

How does the constitutive law look as seen from the original laboratory frame? For this purpose we will use the results of Sec. D.4.4 and the relations between the two foliations established in the previous section.

To begin with, recall of how the excitation and the field strength 2-forms decompose with respect to the the *laboratory frame*

$$I\!H = -\mathfrak{H}\wedge d\sigma + \mathfrak{D}, \qquad F = E\wedge d\sigma + B, \tag{E.4.14}$$ HFlab

and, analogously, with respect to the *material frame*:

$$I\!H = -\mathfrak{H}'\wedge d\tau + \mathfrak{D}', \qquad F = E'\wedge d\tau + B'. \tag{E.4.15}$$ HFmat

Clearly, we preserve the same symbols $I\!H$ and F on the left-hand sides of (E.4.14) and (E.4.15) because these are just the same physical objects. In contrast, the right-hand sides are of course different, hence the primes. The constitutive law (E.4.13), according to the results of the previous section, can be rewritten as

$$\mathfrak{D}' = \varepsilon\varepsilon_0\,\hat{*}E', \qquad \mathfrak{H}' = \frac{1}{\mu\mu_0}\,\hat{*}B'. \tag{E.4.16}$$ const5b

Here the Hodge star $\hat{*}$ corresponds to the metric $\widehat{g}_{ab}$ of the material foliation [please do not mix it up with the Hodge star $\underline{*}$ defined by the 3-space metric ${}^{(3)}g_{ab}$ of the laboratory foliation]. Now, (E.4.16) can be presented in the equivalent matrix form

$$\begin{pmatrix} \mathfrak{H}'_a \\ \mathfrak{D}'^a \end{pmatrix} = \lambda \begin{pmatrix} 0 & \frac{c}{n}\frac{\gamma}{\sqrt{g}}\,\widehat{g}_{ab} \\ -\frac{n}{c}\frac{\sqrt{g}}{\gamma}\,\widehat{g}^{ab} & 0 \end{pmatrix} \begin{pmatrix} -E'_b \\ B'^b \end{pmatrix}. \tag{E.4.17}$$ const5c

The components of the constitutive matrices read explicitly

$$A'^{ab} = -\frac{n}{c}\frac{\sqrt{g}}{\gamma}\,\widehat{g}^{ab}, \quad B'_{ab} = \frac{c}{n}\frac{\gamma}{\sqrt{g}}\,\widehat{g}_{ab}, \quad C'^{a}{}_{b} = 0, \qquad \text{(E.4.18)}$$ ABCfm

$$\text{with}\quad \lambda = \sqrt{\frac{\varepsilon\varepsilon_0}{\mu\mu_0}}, \qquad n := \sqrt{\mu\varepsilon}\,. \qquad \text{(E.4.19)}$$ fem

In order to find the constitutive law in the laboratory frame, we have to perform some very straightforward matrix algebra manipulations along the lines described in Sec. D.4.4. Given is the linear transformation of the coframes (E.4.7). The corresponding transformation of the 2-form basis (A.1.97) turns out to be

$$\begin{aligned} P^{a}{}_{b} &= \frac{\gamma c}{N}\left(\delta^a_b - \frac{1}{c^2}\,v^a v_b\right), \quad Q_b{}^a = \delta^a_b, \\ Z_{ab} &= -\,\gamma\hat{\epsilon}_{abc}\,v^c, \quad W^{ab} = \frac{1}{Nc}\,\epsilon^{abc}\,v_c. \end{aligned} \qquad \text{(E.4.20)}$$

We use these results in (D.4.27)-(D.4.30). Then, after a lengthy matrix computation, we obtain from (E.4.18) the constitutive matrices in the laboratory foliation:

$$A^{ab} = \frac{1}{1-\frac{v^2}{c^2}}\,\frac{\sqrt{{}^{(3)}g}}{N}\left[{}^{(3)}g^{ab}\left(\frac{v^2}{c^2 n} - n\right) + \frac{1}{c^2}\,v^a v^b\left(n - \frac{1}{n}\right)\right], \qquad \text{(E.4.21)}$$ Aem

$$B_{ab} = \frac{1}{1-\frac{v^2}{c^2}}\,\frac{N}{\sqrt{{}^{(3)}g}}\left[{}^{(3)}g_{ab}\left(\frac{1}{n} - \frac{v^2 n}{c^2}\right) + \frac{1}{c^2}\,v_a v_b\left(n - \frac{1}{n}\right)\right], \qquad \text{(E.4.22)}$$ Bem

$$C^{a}{}_{b} = \frac{1}{1-\frac{v^2}{c^2}}\left(n - \frac{1}{n}\right){}^{(3)}\eta^{ac}{}_{b}\,\frac{v_c}{c}. \qquad \text{(E.4.23)}$$ Cem

The resulting constitutive law

$$\begin{pmatrix} \mathfrak{H}_a \\ \mathfrak{D}^a \end{pmatrix} = \lambda \begin{pmatrix} C^{b}{}_{a} & B_{ab} \\ A^{ab} & C^{a}{}_{b} \end{pmatrix} \begin{pmatrix} -E_b \\ B^b \end{pmatrix} \qquad \text{(E.4.24)}$$ CMmoving

can be presented in terms of exterior forms as:

$$\begin{aligned}\mathfrak{H} =&\frac{\gamma^2}{\mu_0\mu_{\rm g}}\left[{}^{\underline{*}}B\left(\frac{1}{\mu}-\varepsilon\,\frac{v^2}{c^2}\right)+\frac{1}{c^2}\,v\;{}^{\underline{*}}(v\wedge B)\left(\varepsilon-\frac{1}{\mu}\right)\right]\\ &+\,\gamma^2\,\varepsilon_0\,{}^{\underline{*}}(v\wedge E)\left(\varepsilon-\frac{1}{\mu}\right), \qquad\qquad \text{(E.4.25)}\end{aligned}$$

Hmov

$$\begin{aligned}\mathfrak{D} =&\gamma^2\varepsilon_0\varepsilon_{\rm g}\left[{}^{\underline{*}}E\left(\varepsilon-\frac{v^2}{\mu\,c^2}\right)-\frac{1}{c^2}\,{}^{\underline{*}}v\;{}^{\underline{*}}(v\wedge{}^{\underline{*}}E)\left(\varepsilon-\frac{1}{\mu}\right)\right]\\ &+\,\gamma^2\,\varepsilon_0\,v\wedge{}^{\underline{*}}B\left(\varepsilon-\frac{1}{\mu}\right). \qquad\qquad \text{(E.4.26)}\end{aligned}$$

Dmov

Here we introduced the 3-velocity 1-form

$$v := v_a\,dx^a. \qquad \text{(E.4.27)}$$

3velocity

The 3-velocity vector is decomposed according to $v^a\,e_a$. If we lower the index v^a by means of the 3-metric ${}^{(3)}g_{ab}$, we find the covariant components v_a of the 3-velocity which enter (E.4.27). The direct inspection shows that the constitutive law (E.4.25)-(E.4.26) of above can alternatively be recasted into the pair of equations:

$$\mathfrak{D}+\frac{\mu_{\rm g}}{c^2}\,v\wedge\mathfrak{H}=\varepsilon\varepsilon_0\,\left(\varepsilon_{\rm g}\,{}^{\underline{*}}E+v\wedge{}^{\underline{*}}B\right), \qquad \text{(E.4.28)}$$

mov1ex

$$B-\frac{\varepsilon_{\rm g}}{c^2}\,v\wedge E=\mu\mu_0\,\left(\mu_{\rm g}\,{}^{\underline{*}}\mathfrak{H}-v\wedge{}^{\underline{*}}\mathfrak{D}\right). \qquad \text{(E.4.29)}$$

mov2ex

These are the famous Minkowski relations for the fields in a moving medium[1]. Originally, the constitutive relations (E.4.28)-(E.4.29) were derived by Minkowski with the help of the Lorentz transformations for the case of a flat spacetime and a uniformly moving media. We stress, however, that the Lorentz group never entered the scene in our above derivation. This demonstrates (contrary to the traditional view) that the role and the value of the Lorentz invariance in electrodynamics should not be overestimated.

The constitutive law (E.4.25)-(E.4.26) or, equivalently, (E.4.28)-(E.4.29), describes a moving simple medium on an *arbitrary curved background*. The influence of the spacetime geometry is manifest in $\varepsilon_{\rm g},\mu_{\rm g}$ and in ${}^{(3)}g_{ab}$ which enters the Hodge star operator. In flat Minkowski spacetime in Cartesian coordinates, these quantities reduce to $\varepsilon_{\rm g}=\mu_{\rm g}=1,{}^{(3)}g_{ab}=\delta_{ab}$.

---

[1] See the discussions of various aspects of electrodynamics of moving media in [2, 18, 19, 4, 13].

The physical sources of the electric and magnetic excitations $\mathfrak{D}$ and $\mathfrak{H}$ are the free charges and currents. Recalling the definitions (E.3.8) and (E.3.5), we can find the polarization P and the magnetization M which have the bound charges and currents as their sources. A direct substitution of (E.4.25) and (E.4.26) into (E.3.8) yields:

$$\begin{aligned} P =&\gamma^2\,\varepsilon_0\varepsilon_{\rm g}\left\{\chi_{\rm E}\left[{}^{\underline{*}}E - \frac{1}{c^2}\,{}^{\underline{*}}v\,{}^{\underline{*}}(v\wedge{}^{\underline{*}}E) + \frac{1}{\mu_{\rm g}}\,v\wedge{}^{\underline{*}}B\right]\right.\\ &\left. + \,\chi_{\rm B}\left[\frac{v^2}{c^2}\,{}^{\underline{*}}E - \frac{1}{c^2}\,{}^{\underline{*}}v\,{}^{\underline{*}}(v\wedge{}^{\underline{*}}E) + \frac{1}{\mu_{\rm g}}\,v\wedge{}^{\underline{*}}B\right]\right\}, \qquad \text{(E.4.30)} \end{aligned}$$

Pmov

$$\begin{aligned} M =&\frac{\gamma^2}{\mu_0\mu_{\rm g}}\left\{\chi_{\rm B}\left[{}^{\underline{*}}B - \frac{1}{c^2}\,v\,{}^{\underline{*}}(v\wedge B) - \frac{\varepsilon_{\rm g}}{c^2}\,{}^{\underline{*}}(v\wedge E)\right]\right.\\ &\left. + \,\chi_{\rm E}\left[\frac{v^2}{c^2}\,{}^{\underline{*}}B - \frac{1}{c^2}\,v\,{}^{\underline{*}}(v\wedge B) - \frac{\varepsilon_{\rm g}}{c^2}\,{}^{\underline{*}}(v\wedge E)\right]\right\}. \qquad \text{(E.4.31)} \end{aligned}$$

Mmov

Here $\chi_{\rm E}$ and $\chi_{\rm B}$ are the electric and magnetic susceptibilities (E.3.15). When the matter is at rest, i.e. $v = 0$, the equations (E.4.30)-(E.4.31) reduce to the rest frame relations (E.3.14).

E.4.3 Optical metric from the constitutive law

A direct check shows that the constitutive matrices (E.4.21)-(E.4.23) satisfy the closure relation (D.2.6), (D.3.17)-(D.3.19). Consequently, a metric of Lorentzian signature is induced by the constitutive law (E.4.24).

The general reconstruction of a metric from a linear constitutive law is given by (D.4.9). Starting from (E.4.18), we immediately find the induced metric in the material foliation:

$$g'^{\rm opt}_{ij} = \sqrt{\frac{n}{c}\frac{\gamma}{\sqrt{g}}}\left(\begin{array}{c|c} \frac{c^2}{n^2} & 0 \\ \hline 0 & -\,\widehat{g}_{ab} \end{array}\right). \qquad \text{(E.4.32)}$$

gijoptfm

Making use of the relation (E.4.7) between the foliations and of the covariance properties proven in Sec. D.4.4, we find the explicit form of the induced metric in the laboratory foliation:

$$g_{ij}^{\rm opt} = \left(\frac{\varepsilon\mu}{-\det g}\right)^{\frac{1}{4}} \left\{g_{ij} - \left(1 - \frac{1}{\varepsilon\mu}\right)\frac{u_i u_j}{c^2}\right\}. \qquad \text{(E.4.33)}$$

gijopt

Here g_{ij} are the components of the metric tensor of spacetime, and u_i are the covariant components of the matter 4-velocity (E.4.10). Note that $g^{ij}\, u_i\, u_j = c^2$, as usual. The contravariant induced metric reads:

$$g^{{\rm opt}\, ij} = \left(\frac{-\det g}{\varepsilon\mu}\right)^{\frac{1}{4}} \left\{g^{ij} - (1 - \varepsilon\mu)\,\frac{u^i u^j}{c^2}\right\}. \qquad \text{(E.4.34)}$$

Such an induced metric $g_{ij}^{\rm opt}$ is usually called the *optical metric* in order to distinguish it from the *true* spacetime metric g_{ij}. It describes the "dragging of the aether" ("Mitführung des Äthers"[2]). The adjective "optical" expresses the fact that all the optical effects in moving matter are determined by the Fresnel equation (D.1.55) which reduces to the equation for the light cone determined by the metric (E.4.33) in the present case.

The nontrivial polarization/magnetization properties of matter are manifestly present even when the medium in the laboratory frame is at rest. Let us consider Minkowski spacetime with $g_{ij} = \mathrm{diag}(c^2, -1, -1, -1)$, for example, and a medium at rest in it. Then $u = \partial_t$ or, in components, $u^i = (1, 0, 0, 0)$. We substitute this into (E.4.33) and find the optical metric

$$g_{ij}^{\rm opt} = \sqrt{\frac{n}{c}}\left(\begin{array}{c|c} \frac{c^2}{n^2} & 0 \\ \hline 0 & -\,\delta_{ab}\end{array}\right). \qquad \text{(E.4.35)}$$

gijoptM

Evidently, the velocity of light c is replaced by c/n with n as the refractive index of the dielectric and magnetic media.

E.4.4 Electromagnetic field generated in moving continua

Let us consider an explicit example which demonstrates the power of the generally covariant constitutive law.

[2]See Gordon [7].

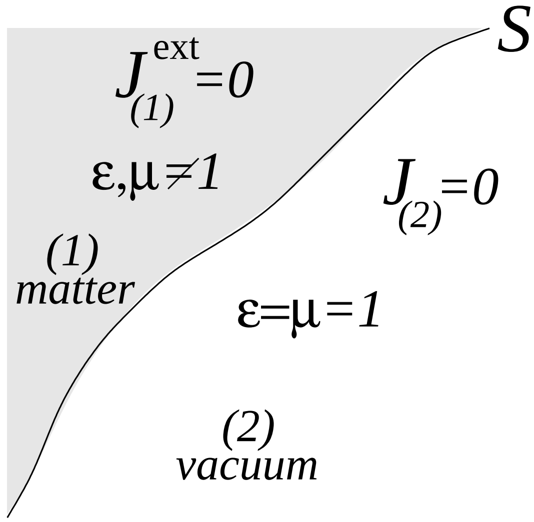

Figure E.4.1: Two regions divided by a surface S.

For simplicity, we will study electrodynamics in flat Minkowski spacetime in which the laboratory reference frame is determined by the usual time coordinate t and the Cartesian spatial coordinates $\vec{x}$. Then the metric has the components $N = c, {}^{(3)}g_{ab} = \delta_{ab}$. Correspondingly, $\varepsilon_{\rm g} = \mu_{\rm g} = 1$. The motion of matter will be, as shown in Sec. E.4.1, described by the material foliation as specified by the relative velocity v^a.

Next, let a surface S be the border between the two regions, the first of which [labeled as (1)] is filled *with matter* having nontrivial magnetic and electric properties with $\mu \neq 1, \varepsilon \neq 1$. In the second region [labeled as (2)], matter is absent, and hence $\mu = \varepsilon = 1$.

We will assume that the matter in the first region does not contain any free (i.e., external) charges and currents, and that the motion of the medium is *stationary*. Then all the variables are independent of time.

Consider the case when the second (matter-free) region contains the *constant* magnetic and/or electric fields, i.e., the components $B^a_{(2)}$ and $E^{(2)}_a$ of the field strengths forms

$$\begin{aligned} B_{(2)} &= B^1\, dx^2 \wedge dx^3 + B^2\, dx^3 \wedge dx^1 + B^3\, dx^1 \wedge dx^2, \\ E_{(2)} &= E_1\, dx^1 + E_2\, dx^2 + E_3\, dx^3, \end{aligned} \tag{E.4.36}$$

BEreg2

do not depend on $t, \vec{x}$. The material law (E.4.24)-(E.4.26) then yields that the components $\mathfrak{H}^{(2)}_a$ and $\mathfrak{D}^a_{(2)}$ of the magnetic and electric excitations forms

$$\begin{aligned} \mathfrak{D}_{(2)} &= \varepsilon_0 \left(E^1\, dx^2 \wedge dx^3 + E^2\, dx^3 \wedge dx^1 + E^3\, dx^1 \wedge dx^2 \right), \\ \mathfrak{H}_{(2)} &= \frac{1}{\mu_0} \left(B_1\, dx^1 + B_2\, dx^2 + B_3\, dx^3 \right), \end{aligned} \tag{E.4.37}$$

DHreg2

are also constant in space and time. The spatial indices are raised and lowered by the spatial metric ${}^{(3)}g_{ab} = \delta_{ab}$. These assumptions evidently guarantee that both the homogeneous $dF = 0$ and inhomogeneous $d\mathbb{H} = J$ Maxwell equations are satisfied for the trivial sources $J = 0$ ($\rho = 0$ and $j = 0$) in the second region.

Let us now verify that the motion of matter generates *non-trivial* electric and magnetic fields in the first region. In order to find their configurations, it is necessary to use the constitutive law (E.4.24)-(E.4.26) and the boundary conditions at the surface S. Recalling the jump conditions on the separating surface (B.4.13)-(B.4.14) and (B.4.15)-(B.4.16), we find, in the absence of free charges and currents:

$$\tau_A \lrcorner\, \mathfrak{H}_{(1)}\Big|_S = \tau_A \lrcorner\, \mathfrak{H}_{(2)}\Big|_S, \qquad \nu \wedge \mathfrak{D}_{(1)}\Big|_S = \nu \wedge \mathfrak{D}_{(2)}\Big|_S, \tag{E.4.38}$$

HDonS

$$\tau_A \lrcorner\, E_{(1)}\Big|_S = \tau_A \lrcorner\, E_{(2)}\Big|_S, \qquad \nu \wedge B_{(1)}\Big|_S = \nu \wedge B_{(2)}\Big|_S. \tag{E.4.39}$$

EBonS

Since the matter is confined to the first region, we conclude that the 3-velocity vector on the boundary surface S has only two tangential components:

$$v^a \partial_a\Big|_S = v^A\, \tau_A, \qquad A = 1, 2. \tag{E.4.40}$$

Let us assume that the two tangential vectors are mutually orthogonal and have unit length (which is always possible to achieve by the suitable choice of the variables ξ^A parametrizing the boundary surface).

The solution of the Maxwell equations $dF = 0$ and $d\mathbb{H} = 0$ in the second region is uniquely defined by the continuity conditions (E.4.38)-(E.4.39). Let us write them down explicitly. Applying $\tau_A \lrcorner$ to (E.4.25) and $\nu\wedge$ to (E.4.26), we find:

$$\begin{aligned}\tau_A \lrcorner \mathfrak{H} =& \frac{\gamma^2}{\mu_0\mu_{\rm g}}\left[\left(\frac{1}{\mu} - \varepsilon\frac{v^2}{c^2}\right)\delta_A^B + \left(\varepsilon - \frac{1}{\mu}\right)\frac{1}{c^2}\,v_A\,v^B\right]\tau_B \lrcorner \underline{*}B\\ &+ \gamma^2\varepsilon_0\left(\varepsilon - \frac{1}{\mu}\right)\epsilon_{AB}\,v^B\,\underline{*}(\nu\wedge\underline{*}E),\end{aligned} \tag{E.4.41}$$

tauHB

$$\begin{aligned}\nu\wedge\mathfrak{D} =& \gamma^2\varepsilon_0\varepsilon_{\rm g}\left(\varepsilon - \frac{v^2}{\mu c^2}\right)\nu\wedge\underline{*}E\\ &+ \gamma^2\varepsilon_0\left(\varepsilon - \frac{1}{\mu}\right)\epsilon^{AB}\,v_A\,\underline{*}(\tau_B \lrcorner \underline{*}B).\end{aligned} \tag{E.4.42}$$

nuDE

Here $\epsilon_{AB} = -\epsilon_{BA}$ with $\epsilon_{\dot{1}\dot{2}} = 1$ (and the same for ϵ^{AB}). These equations should be taken on the boundary surface S. A simple but rather lengthy calculation yields the inverse relations:

$$\begin{aligned}\tau_A \lrcorner \underline{*}B =& \gamma^2\mu_0\mu_{\rm g}\left[\left(\mu - \frac{v^2}{\varepsilon c^2}\right)\delta_A^B - \left(\mu - \frac{1}{\varepsilon}\right)\frac{1}{c^2}\,v_A\,v^B\right]\tau_B \lrcorner \mathfrak{H}\\ &- \gamma^2\mu_0\left(\mu - \frac{1}{\varepsilon}\right)\epsilon_{AB}\,v^B\,\underline{*}(\nu\wedge\mathfrak{D}),\end{aligned} \tag{E.4.43}$$

tauBH

$$\begin{aligned}\nu\wedge\underline{*}E =& \frac{\gamma^2}{\varepsilon_0\varepsilon_{\rm g}}\left(\frac{1}{\varepsilon} - \mu\frac{v^2}{c^2}\right)\nu\wedge\mathfrak{D}\\ &- \gamma^2\mu_0\left(\mu - \frac{1}{\varepsilon}\right)\epsilon^{AB}\,v_A\,\underline{*}(\tau_B \lrcorner \mathfrak{H}).\end{aligned} \tag{E.4.44}$$

nuED

The 3 equations (E.4.43)-(E.4.44), taken on S, together with the 3 equations (E.4.39) are specifying all 6 components of the electric and magnetic field strength E and B on the boundary S in terms of the constant values of the field strengths (E.4.36) in the matter free region.

The standard way to find the static electromagnetic fields in region 1 is as follows. The [(1 + 3)-decomposed] homogeneous Maxwell equations $dE_{(1)} = 0, dB_{(1)} = 0$ are solved by

$E_{(1)} = d\varphi$, $B_{(1)} = d\mathcal{A}$. Substituting this, by using the constitutive law (E.4.25)-(E.4.26), into the inhomogeneous Maxwell equations $d\mathfrak{H}_{(1)} = 0, d\mathfrak{D}_{(1)} = 0$, we obtain the 4 second order differential equations for the 4 independent components of the electromagnetic potential $\varphi(x), \mathcal{A}(x)$. The unique solution of the resulting partial differential system is determined by the boundary conditions (E.4.39) and (E.4.43)-(E.4.44).

E.4.5 The experiments of Röntgen and Wilson & Wilson

In general case, this is a highly nontrivial problem. However, there are two physically important special cases for which the solution is straightforward. They describe the experiments of Röntgen and Wilsons with moving dielectric bodies. In both case we will confine our attention to the choice of the boundary as a plane $S = \{x^3 = 0\}$, so that the tangential vectors and the normal 1-form are:

$$\tau_1 = \partial_1, \qquad \tau_2 = \partial_2; \qquad \nu = dx^3. \tag{E.4.45}$$

We will assume that the upper half-space (corresponding to the positive x^3) is filled with the matter moving with the horizontal velocity

$$v = v_1\, dx^1 + v_2\, dx^2. \tag{E.4.46}$$

Röntgen experiment

Let us consider the case when the magnetic field is absent in the matter-free region, whereas *electric field* is directed towards the boundary:

$$B_{(2)} = 0, \qquad E_{(2)} = E_3\, dx^3. \tag{E.4.47}$$

Then from (E.4.37) we have

$$\mathfrak{H}_{(2)} = 0, \qquad \mathfrak{D}_{(2)} = \varepsilon_0\, E^3\, dx^1 \wedge dx^2. \tag{E.4.48}$$

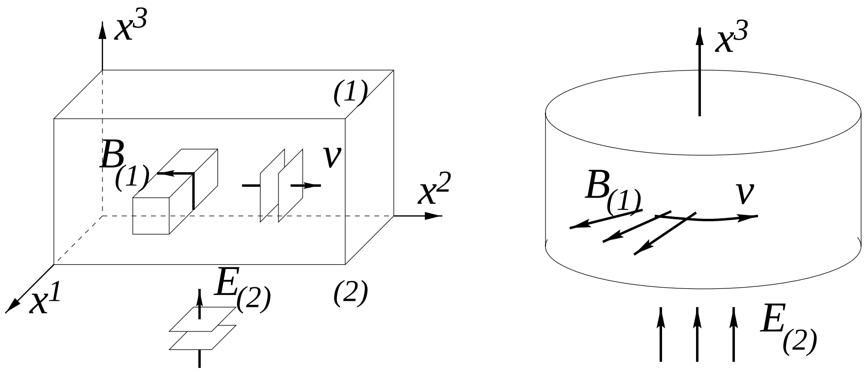

Figure E.4.2: Experiment of Röntgen

It is straightforward to verify that, for the uniform motion (with constant v), the forms

$$B_{(1)} = \frac{1}{1-\frac{v^2}{c^2}}\left(\frac{1}{\varepsilon}-\mu\right)\frac{1}{c^2}\, v\wedge E_3\, dx^3, \tag{E.4.49}$$ Broent

$$E_{(1)} = \frac{1}{1-\frac{v^2}{c^2}}\left(\frac{1}{\varepsilon}-\mu\frac{v^2}{c^2}\right) E_3\, dx^3 \tag{E.4.50}$$ Eroent

describe the solution of the Maxwell equations satisfying the boundary conditions (E.4.39) and (E.4.43)-(E.4.44). Using the constitutive law (E.4.25)-(E.4.26), we find from (E.4.49) and (E.4.50) the corresponding excitation forms:

$$\mathfrak{H}_{(1)} = 0, \qquad \mathfrak{D}_{(1)} = \varepsilon_0\, E^3\, dx^1\wedge dx^2. \tag{E.4.51}$$

This situation is described on left part of the Figure E.4.2: The magnetic field is generated along the x^1 axis by the motion of the matter along the x^2 axis.

In order to simplify the derivations, here we have studied the case of the uniform translational motion of matter. However, in the actual experiment of Röntgen[3] in 1888 he observed this effect for a rotating dielectric disc, as shown schematically on the right part of Figure E.4.2. One can immediately see that in the

[3]See Röntgen [20] and the later thorough experimental study of Eichenwald [5].

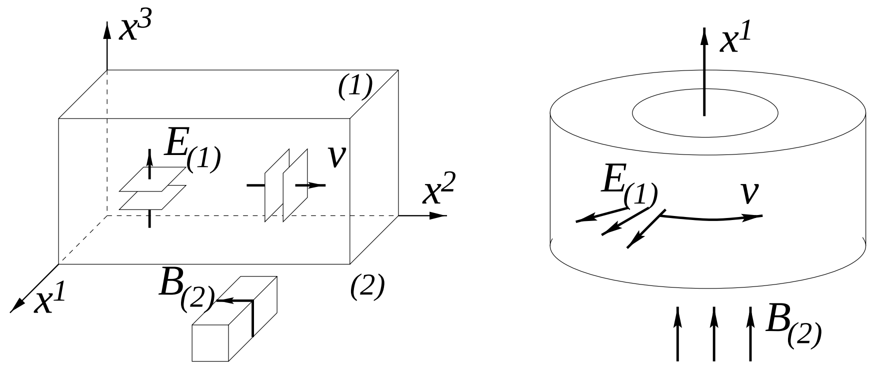

Figure E.4.3: Experiment of Wilson and Wilson

non-relativistic approximation (neglecting the terms v^2/c^2), the formulas (E.4.49)-(E.4.50) describe the solution of the Maxwell equations, provided $dv = 0$. This includes, in particular, the case of the uniform rotation $v = \omega\, d\phi$ with $\omega =$const. Here, ϕ is the usual polar angle, i.e., $d\phi = (x^1 dx^2 - x^2 dx^1)/((x^1)^2 + (x^2)^2)$. The magnetic field generated along the radial direction can be detected by means of a magnetic needle, for example.

Wilson and Wilson experiment

In the 'dual' case the electric field is absent in the matter-free region whereas a magnetic field is pointing along the boundary:

$$B_{(2)} = B^1\, dx^2 \wedge dx^3 + B^2\, dx^3 \wedge dx^1, \qquad E_{(2)} = 0. \quad \text{(E.4.52)}$$

Then from (E.4.37) we find

$$\mathfrak{H}_{(2)} = \frac{1}{\mu_0}\left(B_1\, dx^1 + B_2\, dx^2\right), \qquad \mathfrak{D}_{(2)} = 0. \qquad \text{(E.4.53)}$$

The solution of the Maxwell equations satisfying the boundary conditions (E.4.39), (E.4.43)-(E.4.44) is straightforwardly obtained for uniform motion of the matter:

$$E_{(1)} = \frac{1}{1-\frac{v^2}{c^2}}\left(\frac{1}{\varepsilon}-\mu\right)(v^1 B^2 - v^2 B^1)\, dx^3, \qquad \text{(E.4.54)}$$ Ewils

$$\begin{aligned} B_{(1)} = {} & \mu\,(B^1\, dx^2 \wedge dx^3 + B^2\, dx^3 \wedge dx^1) \\ & + \frac{1}{1-\frac{v^2}{c^2}}\left(\frac{1}{\varepsilon}-\mu\right)\frac{1}{c^2}\, v \wedge (v^1 B^2 - v^2 B^1)\, dx^3. \end{aligned} \qquad \text{(E.4.55)}$$ Bwils

This situation is depicted on the left part of Figure E.4.3. There, without restricting generality, we have chosen the velocity along x^2 and the magnetic field $B_{(2)}$ along x^1. Then the generated electric field is directed along the x^3 axis.

The electric and magnetic excitations in matter are obtained from the constitutive law (E.4.25)-(E.4.26) which, for (E.4.54) and (E.4.55), yields

$$\mathfrak{H}_{(1)} = \frac{1}{\mu_0}\left(B_1\, dx^1 + B_2\, dx^2\right), \qquad \mathfrak{D}_{(1)} = 0. \qquad \text{(E.4.56)}$$

Similarly to the experiment of Röntgen, the experiment of Wilson & Wilson[4] was actually performed for the rotating matter and not for the uniform translational motion described above. The true scheme of the experiment is given on the right side of Figure E.4.3. In fact, the rotating cylinder is formally obtained from the left figure by identifying $x^1 = z$, $x^2 = \phi$, $x^3 = \rho$ with the standard cylindrical coordinates (polar angle ϕ, radius ρ). Usually, one should be careful about the use of curvilinear coordinates in which the components of metric are nonconstant. However, the use of exterior calculus makes all computations transparent and simple. We leave it as an exercise to the reader to verify that the Maxwell equations yield the following exact solution for the cylindrical configuration of the Wilsons experi-

[4] See Wilson and Wilson [23].

ment:

$$\mathfrak{H}_{(2)} = \frac{1}{\mu_0} B\, dz, \qquad \mathfrak{D}_{(2)} = 0, \tag{E.4.57}$$
HD2wilson

$$B_{(2)} = B\, d\rho \wedge \rho\, d\phi, \qquad E_{(2)} = 0, \tag{E.4.58}$$
BE2wilson

$$\mathfrak{H}_{(1)} = \frac{1}{\mu_0} B\, dz, \qquad \mathfrak{D}_{(1)} = 0, \tag{E.4.59}$$

$$B_{(1)} = \frac{1}{1 - \frac{(\omega\rho)^2}{c^2}} \left(\mu - \frac{(\omega\rho)^2}{\varepsilon c^2} \right) B\, d\rho \wedge \rho\, d\phi, \tag{E.4.60}$$
Bwilson

$$E_{(1)} = \frac{1}{1 - \frac{(\omega\rho)^2}{c^2}} \left(\frac{1}{\varepsilon} - \mu \right) \omega\rho\, B\, d\rho. \tag{E.4.61}$$
Ewilson

The boundary conditions (E.4.39), (E.4.43)-(E.4.44) are satisfied for (E.4.57)-(E.4.61). Note that now $\nu = d\rho$ and $\tau_1 = \partial_z, \tau_2 = \partial_\phi$ and the velocity one-form reads

$$v = \omega\rho^2 d\phi, \tag{E.4.62}$$

with constant angular velocity ω. The radial electric field (E.4.61) generated in the rotating cylinder can be detected by measuring the voltage between the inner and the outer surfaces of the cylinder.

One may wonder what physical source is behind the electric and magnetic fields that are generated in moving matter. After all, we have assumed that there are no free charges and currents inside region 1. However, we have bound charges and currents therein described by the polarization and magnetization (E.4.30) and (E.4.31). Substituting (E.4.60)-(E.4.61) into (E.4.30)-(E.4.31), we find:

$$P = \frac{\varepsilon_0}{1 - \frac{(\omega\rho)^2}{c^2}} \left(\mu - \frac{1}{\varepsilon} \right) \omega\rho^2\, B\, d\phi \wedge dz, \tag{E.4.63}$$

$$M = \frac{\varepsilon_0}{1 - \frac{(\omega\rho)^2}{c^2}} \left[\mu - 1 + \frac{(\omega\rho)^2}{c^2} \left(1 - \frac{1}{\varepsilon} \right) \right] c^2\, B\, dz. \tag{E.4.64}$$

From the definition of these quantities (E.3.5), we obtain, merely by taking the exterior differential, the charge and current densities:

$$\rho^{\rm mat} = \frac{2\varepsilon_0}{\left(1-\frac{(\omega\rho)^2}{c^2}\right)^2}\left(\frac{1}{\varepsilon}-\mu\right)\omega\, B\, d\rho\wedge\rho d\phi\wedge dz, \qquad \text{(E.4.65)}$$ rhomov

$$j^{\rm mat} = \frac{2\varepsilon_0}{\left(1-\frac{(\omega\rho)^2}{c^2}\right)^2}\left(\frac{1}{\varepsilon}-\mu\right)\omega^2\rho\, B\, dz\wedge d\rho. \qquad \text{(E.4.66)}$$ jmov

It is these charge and current densities which generate the nontrivial electric and magnetic fields in the rotating cylinder in the Wilsons experiment. The bound current and charge density (E.4.65)-(E.4.66) satisfy the relation

$$j^{\rm mat} = \left({}^{\underline{*}}\rho^{\rm mat}\right){}^{\underline{*}}v. \qquad \text{(E.4.67)}$$

E.4.6 Non-inertial “rotating coordinates”

How is the Maxwell-Lorentz electrodynamics seen by a non-inertial observer? We need a procedure of two steps for the installation of such an observer. In this section the first step is done by introducing suitable non-inertial coordinates.

We assume the absence of a gravitational field. Then spacetime is Minkowskian and a global Cartesian coordinate system t, x^a (with $a = 1, 2, 3$) can be introduced which spans the inertial (reference) frame. The spacetime interval reads

$$ds^2 = c^2dt^2 - \delta_{ab}\, dx^a dx^b. \qquad \text{(E.4.68)}$$ metFlat

As usual, the electromagnetic excitation and the field strength are given by

$$I\!\!H = -\mathfrak{H}\wedge dt + \mathfrak{D}, \qquad F = E\wedge dt + B. \qquad \text{(E.4.69)}$$ HFin

Assuming matter to be at rest in the inertial frame, we have the constitutive law

$$\mathfrak{H} = \frac{1}{\mu\mu_0}\,{}^{\underline{*}}B, \qquad \mathfrak{D} = \varepsilon\varepsilon_0\,{}^{\underline{*}}E. \qquad \text{(E.4.70)}$$ matIn1

Equivalently, we have $\lambda = \sqrt{\frac{\varepsilon\varepsilon_0}{\mu\mu_0}}$ and the constitutive matrices

$$A^{ab} = -\,\frac{n}{c}\,\delta^{ab}, \qquad B_{ab} = \frac{c}{n}\,\delta_{ab}, \qquad C^a{}_b = 0. \qquad \text{(E.4.71)}$$

clawM

The corresponding optical metric is given by (E.4.35).

Now we want to introduce non-inertial "rotating coordinates" (t', x'^a) by

$$t = t', \qquad x^a = L_b{}^a\, x'^b, \qquad \text{(E.4.72)}$$

with the 3×3 matrix

$$L_b{}^a = n^a\, n_b + (\delta_b^a - n^a\, n_b)\cos\varphi + \hat{\epsilon}^a{}_{cb}\, n^c\, \sin\varphi\,. \qquad \text{(E.4.73)}$$

rotL

The matrix defines a rotation of an angle $\varphi = \varphi(t)$ around the direction specified by the constant unit vector $\vec{n} = n^a$, with $\delta_{ab} n^a n^b = 1$. The Latin (spatial) indices are raised and lowered by means of the Euclidean metric δ_{ab} and δ^{ab} ($\hat{\epsilon}^a{}_{cb} = \delta^{ad}\hat{\epsilon}_{dcb}$, for example). We put "rotating coordinates" in quotes, since it is strictly speaking the natural frame (dt', dx'^a) attached to the coordinates (t', x'^a) that is rotating with respect to the Cartesian frame.

The electromagnetic two-forms H and F are independent of coordinates. However, their *components* are different in different coordinate systems. In exterior calculus it is easy to find the components of forms: one only needs to substitute the original natural coframe (dt, dx^a) by the transformed one. The straightforward calculation, using (E.4.73), yields

$$\begin{pmatrix} dt \\ dx^a \end{pmatrix} = \left(\begin{array}{c|c} 1 & 0 \\ \hline L_c{}^a\,[\vec{\omega} \times \vec{x}']^c & L_b{}^a \end{array}\right) \begin{pmatrix} dt' \\ dx'^b \end{pmatrix}. \qquad \text{(E.4.74)}$$

dtdx

Here the angular 3-velocity vector is defined by

$$\vec{\omega} := \omega\,\vec{n}, \qquad \omega := \frac{d\varphi}{dt}. \qquad \text{(E.4.75)}$$

Substituting these differentials into (E.4.68), we find the interval in rotating coordinates:

$$\begin{aligned} ds^2 =& c^2(dt')^2\left[1 + (\vec{\omega}\cdot\vec{x}'/c)^2 - (\vec{\omega}\cdot\vec{\omega}/c^2)(\vec{x}'\cdot\vec{x}')\right] \\ &- 2dt'\, d\vec{x}'\cdot[\vec{\omega}\times\vec{x}'] - d\vec{x}'\cdot d\vec{x}'. \end{aligned} \qquad \text{(E.4.76)}$$

metRot

Electromagnetic excitation and the field strength are expanded with respect to the rotating frame as usual:

$$\mathit{I\!H} = -\mathfrak{H}' \wedge dt + \mathfrak{D}', \qquad F = E' \wedge dt + B'. \tag{E.4.77}$$ HFnin1

Note that $dt = dt'$. The constitutive matrices are derived from (E.4.71) with the help of the transformation (D.4.28)-(D.4.30). Given (E.4.74), we find the matrices (A.1.98)-(A.1.99) as

$$P^a{}_b = L^a{}_b, \quad Q_b{}^a = (L^{-1})_b{}^a, \quad W^{ab} = 0, \quad Z_{ab} = (L^{-1})_a{}^c\,\hat\epsilon_{bcd}\,v^d. \tag{E.4.78}$$ PQrot1

Hereafter we use the abbreviation

$$\vec{v} := [\vec{\omega} \times \vec{x}']. \tag{E.4.79}$$ Vrot

We substitute (E.4.78) into (D.4.28)-(D.4.30) and find the constitutive matrices in the rotating natural frame as

$$A'^{ab} = \frac{1}{c}\left[-\,n\,\delta^{ab}\right], \tag{E.4.80}$$ Aem1rota

$$B'_{ab} = c\left[\frac{1}{n}\,\delta_{ab} + n\left(-\,\delta_{ab}\frac{v^2}{c^2} + \frac{1}{c^2}\,v_a v_b\right)\right], \tag{E.4.81}$$ Bem1rota

$$C'^a{}_b = n\,\delta^{ac}\,\hat\epsilon_{cbd}\,\frac{v^d}{c}, \tag{E.4.82}$$ Cem1rota

with $n = \sqrt{\mu\varepsilon}$. These matrices satisfy the algebraic closure relation (D.3.17)-(D.3.19). Thus they define the induced spacetime metric. The latter is obtained from (D.4.25) by using (E.4.35) and (E.4.74):

$$g'^{\rm opt}_{ij} = \sqrt{\frac{n}{c}}\left(\begin{array}{c|c} \frac{c^2}{n^2} - v^2 & -\,v_b \\ \hline -\,v_a & -\,\delta_{ab} \end{array}\right). \tag{E.4.83}$$ gijoptrota

E.4.7 Rotating observer

However, the behavior of fields with respect to rotating frame is usually of minor physical interest to us. The observer rather measures all physical quantities with respect to a local frame

ϑ^α which is anholonomic in general, i.e. $d\vartheta^\alpha \neq 0$. The observer is, in fact, *comoving* with that frame and the components of excitation and field strength should be determined with respect to ϑ^α. Consequently, the observer's 4-velocity vector reads

$$e_{\widehat{0}} = e^i{}_{\widehat{0}}\,\partial_i = \gamma\,\partial_{t'}\,. \qquad \text{(E.4.84)}$$ e0rot

The 'Lorentz' factor

$$\gamma = \frac{1}{\sqrt{1-\vec{v}^2/c^2}}, \qquad \text{(E.4.85)}$$

is determined for the metric (E.4.76) by the normalization condition $\mathbf{g}(e_{\widehat{0}}, e_{\widehat{0}}) = e^i{}_{\widehat{0}}\, e^j{}_{\widehat{0}}\, g'_{ij} = c^2$. Note that $\vec{v}^2 = \delta_{ab}\, v^a\, v^b$.

The observer's rotating frame e_α with (E.4.84) and

$$e_{\widehat{a}} = \partial_{x'^a} + \frac{\gamma^2}{c^2}\, v_a\, \partial_{t'} \qquad \text{(E.4.86)}$$

is dual to the corresponding coframe ϑ^α with

$$\vartheta^{\widehat{0}} = \frac{1}{\gamma}\, dt' - \frac{\gamma}{c^2}\, \vec{v}\cdot d\vec{x}', \qquad \vartheta^{\widehat{a}} = dx'^a. \qquad \text{(E.4.87)}$$ the0rot

Expressed in terms of this coframe, the metric (E.4.76) reads

$$ds^2 = c^2\,(\vartheta^{\widehat{0}})^2 - \left(\delta_{ab} + \frac{\gamma^2}{c^2}\, v_a\, v_b\right) \vartheta^{\widehat{a}} \otimes \vartheta^{\widehat{b}}. \qquad \text{(E.4.88)}$$ metcorot

Combining (E.4.74) with (E.4.87), we obtain the transformation from the inertial (dt, dx^a) coframe to the non-inertial $(\vartheta^{\widehat{0}}, \vartheta^{\widehat{a}})$ frame as

$$\begin{pmatrix} dt \\ dx^a \end{pmatrix} = \left(\begin{array}{c|c} \gamma & \frac{\gamma^2}{c^2}\, v_b \\ \hline \gamma\, L_c{}^a\, v^c & L_c{}^a \left(\delta^c_b + \frac{\gamma^2}{c^2}\, v^c\, v_b\right) \end{array}\right) \begin{pmatrix} \vartheta^{\widehat{0}} \\ \vartheta^{\widehat{b}} \end{pmatrix}. \qquad \text{(E.4.89)}$$ dtdx0

With respect to ϑ^α, the electromagnetic excitation and field strength read

$$I\!\!H = -\mathfrak{H}' \wedge \vartheta^{\widehat{0}} + \mathfrak{D}', \qquad F = E' \wedge \vartheta^{\widehat{0}} + B'. \qquad \text{(E.4.90)}$$ HFnin

Using (E.4.89) in the transformation formulas (D.4.28)-(D.4.30), the constitutive law in the frame of a rotating observer turns out to be defined by the constitutive matrices

$$A'^{ab} = \frac{\gamma}{c}\left[n\left(-\,\delta^{ab} + \frac{1}{c^2}\,v^a v^b\right)\right], \tag{E.4.91}$$
Aem1rot

$$B'_{ab} = \frac{c}{\gamma}\left[\frac{1}{n}\left(\delta_{ab} + \frac{\gamma^2}{c^2}\,v_a v_b\right) + n\left(-\,\delta_{ab}\frac{v^2}{c^2} + \frac{1}{c^2}\,v_a v_b\right)\right], \tag{E.4.92}$$
Bem1rot

$$C'^{a}{}_{b} = n\,\delta^{ac}\,\hat{\epsilon}_{cbd}\,\frac{v^d}{c}. \tag{E.4.93}$$
Cem1rot

The matrices (E.4.91)-(E.4.93) satisfy the algebraic closure relation (D.3.17)-(D.3.19). The corresponding optical metric is obtained from (D.4.25), (E.4.35) and (E.4.89) as

$$g'^{\rm opt}_{ij} = \sqrt{\frac{n}{c\gamma}}\left(\begin{array}{c|c} \frac{c^2}{n^2} - v^2 & -\,\gamma v_b \\ \hline -\,\gamma v_a & -\left(\delta_{ab} + \frac{\gamma^2}{c^2}\,v_a v_b\right)\end{array}\right). \tag{E.4.94}$$
gijoptrot

In exterior calculus, the constitutive law (E.4.91)-(E.4.93) in the rotating frame reads

$$\mathfrak{H}' = \frac{1}{\mu_0}\left(\frac{1}{\mu} - \varepsilon\,\frac{v^2}{c^2}\right){}^{*'}\!B' + \varepsilon\varepsilon_0\left[v'\;{}^{*'}(v' \wedge B') - {}^{*'}(v' \wedge E')\right], \tag{E.4.95}$$

$$\mathfrak{D}' = \varepsilon\varepsilon_0\left({}^{*'}\!E' - v' \wedge {}^{*'}\!B'\right), \tag{E.4.96}$$

Here ${}^{*'}$ denotes the Hodge operator with respect to the corresponding 3-space metric $\delta_{ab} + \frac{\gamma^2}{c^2}\,v_a\,v_b$ [see (E.4.88)], and we introduced the velocity 1-form

$$v' := \gamma\,\delta_{ab}\,v^a\,dx'^b. \tag{E.4.97}$$

E.4.8 Accelerating observer

Let us now analyze the case of pure acceleration. It is quite similar to the pure rotation that was considered in Sec. E.4.6.

More concretely, we will study the motion in a fixed spatial direction with an acceleration 3-vector parametrized as

$$\vec{a} = a\,\vec{n}, \qquad \text{or} \qquad a^b = a\,n^b. \tag{E.4.98}$$ accel0

Here $a^2 := \vec{a}\cdot\vec{a}$ is the magnitude of acceleration, and the unit vector $\vec{n}$, with $\vec{n}\cdot\vec{n} = n^b n_b = 1$, gives its direction in space. Recall that we are in the Minkowski spacetime (E.4.68). The accelerating coordinates (t', x'^a) can be obtained from the Cartesian ones (t, x^a) by means of the transformation

$$t = \frac{1}{c}\,\sinh\phi\, n_a\, x'^a + \int^{t'} d\tau\, \cosh\phi(\tau), \tag{E.4.99}$$ a-0-n

$$x^a = \mathcal{K}_b{}^a\, x'^b + cn^a \int^{t'} d\tau\, \sinh\phi(\tau). \tag{E.4.100}$$ a-i-n

Here we denote the 3×3 matrix

$$\mathcal{K}_b{}^a = (\delta^a_b - n^a n_b) + n^a n_b\, \cosh\phi. \tag{E.4.101}$$ Kij

The scalar function $\phi(t')$ determines the magnitude of the acceleration by

$$a(t') = c\,\frac{d\phi}{dt'}. \tag{E.4.102}$$ accel1

Differentiating (E.4.99)-(E.4.100), we obtain the transformation from the inertial coframe to the accelerating one:

$$\begin{pmatrix} dt \\ dx^a \end{pmatrix} = \left(\begin{array}{c|c} \left(1 + \frac{\vec{a}\cdot\vec{x}'}{c^2}\right)\cosh\phi & \frac{n_b}{c}\sinh\phi \\ \hline \left(1 + \frac{\vec{a}\cdot\vec{x}'}{c^2}\right) cn^a \sinh\phi & \mathcal{K}_b{}^a \end{array}\right) \begin{pmatrix} dt' \\ dx'^b \end{pmatrix}. \tag{E.4.103}$$ dtdx-a-n

Substituting (E.4.103) into (E.4.68), we find the metric in accelerating coordinates:

$$ds^2 = c^2\left(1 + \vec{a}\cdot\vec{x}'/c^2\right)^2 (dt')^2 - d\vec{x}'\cdot d\vec{x}'. \tag{E.4.104}$$ metAcc

This is one of the possible forms of the well known Rindler spacetime.

It is straightforward to construct the local frame of a noninertial observer which is comoving with the accelerating coordinate system. With respect to the original Cartesian coordinates, it reads:

$$e_{\widehat{0}} = \frac{1}{c}\, u, \qquad e_{\widehat{a}} = \frac{n_a}{c}\, \sinh\phi\, \partial_t + \mathcal{K}_a{}^b\, \partial_{x^b}. \tag{E.4.105}$$ frame-acc

Here

$$u = \cosh\phi\, \partial_t + c n^a\, \sinh\phi\, \partial_{x^a} \tag{E.4.106}$$

is the observer's 4-velocity vector field which satisfies $\mathbf{g}(u,u) = c^2$. Clearly, the vectors of the basis (E.4.105), in the sense of the Minkowski 4-metric (E.4.68), are mutually orthogonal and normalized:

$$\mathbf{g}(e_{\widehat{0}}, e_{\widehat{0}}) = 1, \quad \mathbf{g}(e_{\widehat{0}}, e_{\hat{a}}) = 0, \quad \mathbf{g}(e_{\widehat{a}}, e_{\widehat{b}}) = -\,\delta_{ab}. \tag{E.4.107}$$

Because of (E.4.103), the coordinate bases are related by

$$\begin{pmatrix} \partial_t \\ \partial_{x^a} \end{pmatrix} = \left(\begin{array}{c|c} \cosh\phi / \left(1 + \frac{\vec{a}\cdot\vec{x}'}{c^2}\right) & -cn^b \sinh\phi \\ \hline -\frac{n_a}{c} \sinh\phi / \left(1 + \frac{\vec{a}\cdot\vec{x}'}{c^2}\right) & \mathcal{K}_a{}^b \end{array}\right) \begin{pmatrix} \partial_{t'} \\ \partial_{x'^b} \end{pmatrix}. \tag{E.4.108}$$ dtdxinv

Thus, the accelerated frame (E.4.105), with respect to the *accelerating* coordinate system, is described by the simple expressions

$$e_{\widehat{0}} = \frac{1}{c(1 + \vec{a}\cdot\vec{x}/c^2)}\, \partial_{t'}, \qquad e_{\widehat{a}} = \partial_{x'^a}. \tag{E.4.109}$$ frame-nih

According to the definition of the covariant differentiation, see (C.1.15), one has $\nabla_u e_\alpha = \Gamma_\alpha{}^\beta(u) e_\beta$. This enables us to compute the proper time derivatives for the frame (E.4.105), (E.4.109):

$$\dot{u} := \nabla_u\, u = \frac{a^b}{1 + \vec{a}\cdot\vec{x}/c^2}\, e_{\widehat{b}}, \tag{E.4.110}$$ propaccel

$$\dot{e}_{\widehat{b}} := \nabla_u\, e_{\widehat{b}} = \frac{a_b/c^2}{1 + \vec{a}\cdot\vec{x}/c^2}\, u. \tag{E.4.111}$$ dotEa

This means that the frame (E.4.105) is Fermi-Walker transported along the observer's world line.

The coframe dual to the frame (E.4.105), (E.4.109) reads, with respect to the accelerating coordinate system,

$$\vartheta^{\widehat{0}} = \left(1 + \vec{a}\cdot\vec{x}/c^2\right) c dt', \qquad \vartheta^{\widehat{a}} = dx'^a. \tag{E.4.112}$$

cofr-acc

E.4.9 The proper reference frame of the noninertial observer ("noninertial frame")

The line elements (E.4.76) and (E.4.104) of spacetime represent the Minkowski space in rotating and accelerating coordinate systems, respectively. Both are particular cases of the interval

$$\begin{aligned} ds^2 = c^2(dt')^2 \Big[\left(1 + \vec{a}\cdot\vec{x}'/c^2\right)^2 + (\vec{\omega}\cdot\vec{x}'/c)^2 - (\vec{\omega}\cdot\vec{\omega}/c^2)(\vec{x}'\cdot\vec{x}')\Big] \\ - 2dt'\, d\vec{x}'\cdot[\vec{\omega}\times\vec{x}'] - d\vec{x}'\cdot d\vec{x}'. \end{aligned} \tag{E.4.113}$$

met-ni

Here the 3-vectors of acceleration $\vec{a}$ $(= a^b)$ and of angular velocity $\vec{\omega}$ $(= \omega^c)$ can be arbitrary functions of time t'. The interval (E.4.113) reduces to the diagonal form $ds^2 = \vartheta^{\widehat{0}}\otimes\vartheta^{\widehat{0}} - \vartheta^{\widehat{1}}\otimes\vartheta^{\widehat{1}} - \vartheta^{\widehat{2}}\otimes\vartheta^{\widehat{2}} - \vartheta^{\widehat{3}}\otimes\vartheta^{\widehat{3}}$ in the orthonormal coframe[5]:

$$\vartheta^{\widehat{0}} = \left(1 + \vec{a}\cdot\vec{x}/c^2\right)\, cdt', \tag{E.4.114}$$

cofr0

$$\vartheta^{\widehat{a}} = dx'^a + [\vec{\omega}\times\vec{x}']^a\, dt'. \tag{E.4.115}$$

cofri

The corresponding vectors of the dual frame are

$$e_{\widehat{0}} = \frac{1}{c(1 + \vec{a}\cdot\vec{x}'/c^2)}\left(\partial_{t'} - [\vec{\omega}\times\vec{x}']^a\,\partial_{x'^a}\right), \tag{E.4.116}$$

HNI0

$$e_{\widehat{a}} = \partial_{x'^a}. \tag{E.4.117}$$

HNIa

With respect to the local frame chosen, the components $\Gamma_\alpha{}^\beta$ of the Levi-Civita connection read:

$$\Gamma_{\widehat{0}}{}^{\widehat{b}} = \Gamma_{\widehat{b}}{}^{\widehat{0}} = \frac{(a^b/c^2)}{1 + \vec{a}\cdot\vec{x}'/c^2}\,\vartheta^{\widehat{0}} = (a^b/c)\, dt', \tag{E.4.118}$$

GamNI1

$$\Gamma_{\widehat{a}}{}^{\widehat{b}} = \frac{\hat{\epsilon}_{abc}\,(\omega^c/c)}{1 + \vec{a}\cdot\vec{x}'/c^2}\,\vartheta^{\widehat{0}} = \hat{\epsilon}_{abc}\,\omega^c\, dt'. \tag{E.4.119}$$

GamNI2

---

[5] See Hehl and Ni [8].

We can readily check that $d\Gamma_\alpha{}^\beta = 0$ and that the exterior products of the connection 1-forms are zero. As a result, the Riemannian curvature 2-form of the metric (E.4.113) vanishes, $\widetilde{R}_\alpha{}^\beta = 0$. Thus, indeed, we are in a flat spacetime as seen by a non-inertial observer moving with acceleration $\vec{a}$ and angular velocity $\vec{\omega}$.

After these geometrical preliminaries, we can address the problem of how a noninertial observer (accelerating and/or rotating) sees the electrodynamical effects in his proper reference frame (E.4.116), (E.4.117). In order to apply the results of the previous sections, let us specialize either to the case of pure rotation or of pure acceleration.

To begin with, we note that the constitutive relation has its usual form (E.4.70), (E.4.71) in the inertial Cartesian coordinate system (E.4.68). Putting $\vec{a} = 0$, we find from (E.4.114), (E.4.115) the proper coframe of a rotating observer:

$$\vartheta^{\widehat{0}} = dt', \qquad \vartheta^{\widehat{a}} = dx'^a + [\vec{\omega} \times \vec{x}']^a \, dt'. \tag{E.4.120}$$ cofraHN

Combining this with (E.4.74), we find the transformation of the inertial coframe to the non-inertial (rotating) one,

$$\begin{pmatrix} dt \\ dx^a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & L_b{}^a \end{pmatrix} \begin{pmatrix} \vartheta^{\widehat{0}} \\ \vartheta^{\widehat{b}} \end{pmatrix}. \tag{E.4.121}$$

Correspondingly, substituting this into the transformation (D.4.28)-(D.4.30), we immediately find from (E.4.70) the constitutive law in the rotating observer's frame:

$$\mathfrak{H}'_{\widehat{a}} = \frac{1}{\mu\mu_0}\, \delta_{ab}\, B'^{\widehat{a}}, \qquad \mathfrak{D}'^{\widehat{a}} = \varepsilon\varepsilon_0\, \delta^{ab}\, E'_{\widehat{a}}. \tag{E.4.122}$$ constNON

The same result holds true for an accelerating observer. If we put $\vec{\omega} = 0$ in (E.4.114), (E.4.115), we arrive at the coframe (E.4.112). Combined with (E.4.103), this yields the transformation of the inertial coframe to the non-inertial (accelerating) one:

$$\begin{pmatrix} dt \\ dx^a \end{pmatrix} = \left(\begin{array}{c|c} \cosh\phi & \frac{n_b}{c}\sinh\phi \\ \hline cn^a \sinh\phi & \mathcal{K}_b{}^a \end{array}\right) \begin{pmatrix} \vartheta^{\widehat{0}} \\ \vartheta^{\widehat{b}} \end{pmatrix}. \tag{E.4.123}$$

When we use this in (D.4.28)-(D.4.30), the final constitutive law again turns out to be (E.4.122).

Summing up, despite the fact that the proper coframe (E.4.114), (E.4.115) is *non*inertial, the constitutive relation remains in this coframe formally the same as in the inertial coordinate system.[6]

E.4.10 Universality of the Maxwell-Lorentz spacetime relation

The use of foliations and of exterior calculus for the description of the reference frames enables us to establish the universality of the Maxwell-Lorentz spacetime relation.

Let us put $\mu = \varepsilon = 1$ (hence $n = 1$) in the formulas above. Physically, this corresponds to a transformation from one frame (σ-foliation) to another frame (τ-foliation) which moves with an arbitrary velocity u relative to the first one. Then the relation (E.4.13) reduces to

$$\underset{\sim}{I\!H} = \lambda\,(\underset{\sim}{{}^\star F}), \qquad {}^{\dashv}I\!H = \lambda\,{}^{\dashv}({}^\star F). \tag{E.4.124}$$ const6

On the other hand, from (E.4.21)-(E.4.23), we find for the constitutive coefficients:

$$A^{ab} = -\frac{\sqrt{{}^{(3)}g}}{N}\,{}^{(3)}g^{ab}, \quad B_{ab} = \frac{N}{\sqrt{{}^{(3)}g}}\,{}^{(3)}g_{ab}, \quad C^a{}_b = 0. \tag{E.4.125}$$

Equivalently, from (E.4.24) we read off that

$$\mathfrak{H}_a = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,N\,B_a/\sqrt{g}, \qquad \mathfrak{D}^a/\sqrt{g} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{1}{N}\,E^a. \tag{E.4.126}$$

or, returning to exterior forms,

$$\mathfrak{D} = \varepsilon_0\varepsilon_{\rm g}\,\underline{\star}E, \qquad \mathfrak{H} = \frac{1}{\mu_0\mu_{\rm g}}\,\underline{\star}B. \tag{E.4.127}$$ const6b

[6]This shows that it is misleading to associate the "Cartesian form" of a constitutive relation with the inertial frames of reference. Kovetz [13], for example, takes (E.4.122) as a sort of definition of the inertial frames.

This is nothing but a $(1+3)$ decomposition with respect to the laboratory foliation:

$$\underline{I\!H} = \lambda\,\underline{({}^\star F)}, \qquad {}^{\perp}I\!H = \lambda\,{}^{\perp}({}^\star F). \qquad \text{(E.4.128)}$$ const6a

Comparing (E.4.124) with (E.4.128), we arrive at the conclusion that (E.4.124) and (E.4.128) are just different "projections" of the generally valid Maxwell-Lorentz spacetime relation

$$I\!H = \lambda\,{}^*F. \qquad \text{(E.4.129)}$$ HastF

In this form, the Maxwell-Lorentz spacetime relation is valid *always and everywhere*. Neither the choice of coordinates or of a specific reference frame (foliation) play any role. Consequently, our fifth axiom has a universal physical meaning.

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Part F

Preliminary sketch version of

Validity of classical electrodynamics, interaction with gravity, outlook

Of course, electrodynamics describes but one of the *four* interactions in nature. And *classical* electrodynamics only the non-quantum aspects of the electromagnetic field. Therefore electrodynamics relates to the other fields of knowledge in physics in a multitude of different ways. Let us first explore the classical domain.

F.1

Classical physics *(preliminary)*

F.1.1 Gravitational field

The only other interaction, besides the electromagnetic one, that can be described by means of the classical field concept, is the gravitational interaction. Einstein's theory of gravity, general relativity (GR),[1] describes the gravitational field successfully in the macrophysical domain. In GR, spacetime is a 4-dimensional Riemannian manifold with a metric $\mathbf{g}$ of Lorentzian signature. The metric is the gravitational potential. The curvature 2-form $\widetilde{R}_\alpha{}^\beta$, subsuming up to 2nd derivatives of the metric, represents the gravitational field strength. Einstein's field equation, with respect to an arbitrary coframe ϑ^α, reads

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\wedge\widetilde{R}^{\beta\gamma}=\frac{8\pi G}{c^3}\,\overset{\rm Mat}{\sigma}{}_\alpha\;. \qquad \text{(F.1.1)}$$ Einstein

The tilde labels Riemannian objects, G is the Newtonian gravitational constant. The source of the right hand side is the *sym-*

[1]Compare Landau-Lifshitz [16], e.g.. Einstein's Princeton lectures [5] still give a good idea of the underlying principles and some of the main results of GR. Frankel [6] has written a little book on GR underlining its geometrical character and, in particular, developing it in terms of exterior calculus.

metric energy-momentum current of “matter”,

$$\vartheta_{[\alpha} \wedge \overset{\text{Mat}}{\sigma}{}_{\beta]} = 0\,, \tag{F.1.2}$$ Einsym

embodying all non-gravitational contributions to energy.

Electrodynamics fits smoothly into this picture. The Maxwell equations remain the same,

$$dH = J\,, \qquad dF = 0\,. \tag{F.1.3}$$ Einmax

However, the Hodge star in the spacetime relation

$$H = \lambda\,{}^\star F \tag{F.1.4}$$ Einstr

“feels” now the dynamical metric $\mathbf{g}$ fulfilling the Einstein equation.

The electromagnetic field, in the framework of GR, belongs to the matter side,

$$\overset{\text{Mat}}{\sigma}{}_{\alpha} = \overset{\text{mat}}{\sigma}{}_{\alpha} + \overset{\text{Max}}{\Sigma}{}_{\alpha}\;. \tag{F.1.5}$$ Einmatter

In other words, the electromagnetic field enters the gravity scene via its energy-momentum current $\overset{\text{Max}}{\Sigma}{}_{\alpha}$. Since $\vartheta_{[\alpha} \wedge \overset{\text{Max}}{\Sigma}{}_{\beta]} = 0$, also this is possible in a smooth way.

These are the basics of the gravito-electromagnetic complex. Let us illustrate it by an example.

Reissner-Nordström solution

We consider the gravitational and the electromagnetic field in vacuum of a point source of mass m and charge Q. The Einstein equation for this case reads

$$\eta_{\alpha\beta\gamma} \wedge \widetilde{R}^{\beta\gamma} = \frac{8\pi G}{c^3}\,\lambda\,\left[F \wedge (e_\alpha \rfloor {}^\star F) - {}^\star F \wedge (e_\alpha \rfloor F)\right]\,, \tag{F.1.6}$$ Einstein1

for the electromagnetic equations see (F.1.3) and (F.1.4).

For solving such a problem, one will take a spherically symmetric coframe and expresses it in spherical coordinates r, θ, ϕ:

$$\vartheta^{\hat{0}} = f\,cdt\,, \quad \vartheta^{\hat{1}} = \frac{1}{f}\,dr\,, \quad \vartheta^{\hat{2}} = r\,d\theta\,, \quad \vartheta^{\hat{3}} = r\,\sin\theta\,d\phi\,. \tag{F.1.7}$$

coframe1

It contains the zero-form $f = f(r)$ and is assumed to be orthonormal, i.e., the metric reads

$$ds^2 = o_{\alpha\beta}\,\vartheta^\alpha \otimes \vartheta^\beta = f^2 c^2 dt^2 - \frac{1}{f^2} dr^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\,. \tag{F.1.8}$$

metric1

In a Minkowski spacetime, i.e., *without gravity*, we have $f = 1$. The spherically symmetric electromagnetic field is described by the Coulomb type ansatz:

$$A = -\frac{qc}{r}\,dt\,, \tag{F.1.9}$$

Eincoulomb1

$$F = dA = \frac{q}{r^2}\,\vartheta^{\hat{1}} \wedge \vartheta^{\hat{0}}\,. \tag{F.1.10}$$

Eincoulomb2

Here q is a constant. The homogeneous Maxwell equation $dF = 0$ as well as the inhomogeneous equation for the vacuum case $d\,{}^\star F = 0$ are both fulfilled. The energy-momentum current can be determined by a substitution of (F.1.10) into the explicit expression (E.1.23) as

$$\overset{\text{Max}}{\Sigma}{}_\alpha = \lambda\,\frac{q^2}{2r^4}\left(\delta^{\hat{0}}_\alpha\,\eta_{\hat{0}} + \delta^{\hat{1}}_\alpha\,\eta_{\hat{1}} - \delta^{\hat{2}}_\alpha\,\eta_{\hat{2}} - \delta^{\hat{3}}_\alpha\,\eta_{\hat{3}}\right)\,. \tag{F.1.11}$$

maxenergy1

Clearly, Einstein's equations (F.1.1) are *not* fulfilled: The geometric left-hand side vanishes and does not counterbalance the nontrivial right-hand side (F.1.11).

The integration constant q is related to the total charge of the source. From (B.1.1) and (B.1.41) we have, by using the Stokes theorem:

$$Q = \int\limits_{\Omega_3} \rho = \int\limits_{\partial\Omega_3} \mathcal{D}, \tag{F.1.12}$$

totalQ

where the 3-dimensional domain Ω_3 contains the source inside. From (F.1.10) and (F.1.4) we find

$$H = \mathcal{D} = \lambda \frac{q}{r^2} \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} = \lambda\, q \,\sin\theta d\theta \wedge d\phi. \tag{F.1.13}$$

Integration in (F.1.12) is elementary, yielding

$$Q = 4\pi\lambda q. \tag{F.1.14}$$

Recalling that $\lambda = \sqrt{\varepsilon_0/\mu_0}$ and $c = 1/\sqrt{\varepsilon_0\mu_0}$, we then find the standard SI form of the Coulomb potential (F.1.9):

$$A = -\frac{Q}{4\pi\varepsilon_0\, r}\, dt. \tag{F.1.15}$$

Eincoulomb3

Let us now turn to the gravitational case for an electrically *uncharged* sphere. Then, as is known from GR, we have the Schwarzschild solution with

$$f^2 = 1 - \frac{2Gm}{c^2 r}\,. \tag{F.1.16}$$

Einschwarz

Here m is the mass of the source. It is an easy exercise with computer algebra to prove that the vacuum Einstein equation $\frac{1}{2}\,\eta_{\alpha\beta\gamma} \wedge \widetilde{R}^{\beta\gamma} = 0$ is fulfilled for this choice of the coframe.

GR is a nonlinear field theory. Nevertheless, if we now treat the combined case with electromagnetic and gravitational field, we can sort of superimpose the single solutions because of our coordinate and frame invariant presentation of electrodynamics. We have now $f \neq 1$, but still we keep the ansatz for the Coulomb field (F.1.9). The form of the field strength (F.1.10) remains the same in terms of the coframe (F.1.8). Also the energy-momentum current (F.1.11) does not change. Hence we can write down the Einstein field equation (F.1.6) with an explicitly known right hand side. For the unknown function f^2 we can make the ansatz $f^2 = 1 - 2Gm/c^2r + U(r)$. For $U = 0$, we recover the Schwarzschild case. If we substitute this in the left hand side of (F.1.6), then we find (also most conveniently by means of computer algebra) an ordinary differential equation of 2nd order for $U(r)$ which can be easily solved. The result reads:

$$f^2 = 1 - \frac{2Gm}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2} \tag{F.1.17}$$
func1

This, together with the electric field (F.1.15), represents the *Reissner-Nordström* solution of GR for a massive charged "particle".

The electromagnetic field of the Reissner-Nordström solution has the same innocent appearance as that of a point charge in *flat* Minkowski spacetime. It is clear, however, that all relevant geometric objects, coframe, metric, connection, curvature, 'feel' – via the zero-form f – the presence of the electric charge. If the charge satifies the inequality

$$\frac{Q^2}{4\pi\varepsilon_0} \leq Gm^2, \tag{F.1.18}$$
QlessM

then the spacetime metric (F.1.8) has horizons which correspond to the zeros of the function (F.1.17). However, as it is clearly seen from (F.1.9), (F.1.10) and especially (F.1.11), the electromagnetic field is regular everywhere except for the origin. The arising geometry describes a charged *black hole*. When the charge is so large that (F.1.18) becomes invalid, a solution is no black hole but describes a bare singularity.

These results can be straightforwardly generalized to gauge theories of gravity with post-Riemannian pieces in the connection, see [22, 9].

Rotating source: Kerr-Newman solution

When a source is rotating, its electromagnetic and gravitational fields are no longer spherically symmetric. Instead, the Reissner-Nordström geometry of above is replaced by the *axially symmet-*

ric configuration described by the coframe

$$
\begin{aligned}
\vartheta^{\hat 0} &= \sqrt{\frac{\Delta}{\Sigma}}\,\left(c dt - a\sin^2\theta\, d\phi\right),\\
\vartheta^{\hat 1} &= \sqrt{\frac{\Sigma}{\Delta}}\, d r,\\
\vartheta^{\hat 2} &= \sqrt{\Sigma}\, d\theta,\\
\vartheta^{\hat 3} &= \frac{\sin\theta}{\sqrt{\Sigma}}\,\left[-acdt + \left(r^2+a^2\right) d\phi\right],
\end{aligned}
\tag{F.1.19}
$$

frameKerr1

where $\Delta = \Delta(r), \Sigma = \Sigma(r,\theta)$, and a is a constant. The latter is directly related to the angular momentum of the source.

The electromagnetic potential 1-form reads

$$
A = A_{\hat 0}\,\vartheta^{\hat 0}, \tag{F.1.20}
$$

AKerr1

with $A_{\hat 0} = A_{\hat 0}(r,\theta)$. Substituting the ansatz (F.1.19)-(F.1.20) into Einstein-Maxwell field equations (F.1.6), (F.1.3) and (F.1.4), one finds:

$$
\Delta = r^2 + a^2 - \frac{2Gmr}{c^2} + \frac{GQ^2}{4\pi\varepsilon_0 c^4}, \tag{F.1.21}
$$

sol2a

$$
\Sigma = r^2 + a^2\cos^2\theta, \tag{F.1.22}
$$

sol2b

$$
A_{\hat 0} = -\frac{Q}{4\pi\varepsilon_0}\,\frac{r}{\sqrt{\Delta\Sigma}}. \tag{F.1.23}
$$

sol2c

Accordingly, the electromagnetic field strength reads:

$$
\begin{aligned}
F = dA = \frac{Q}{4\pi\varepsilon_0\Sigma^2}\Big(&(a^2\cos^2\theta - r^2)\,\vartheta^{\hat 0}\wedge\vartheta^{\hat 1}\\
&+ \frac{2a^2 r\,\sin\theta\cos\theta}{\sqrt{\Delta}}\,\vartheta^{\hat 0}\wedge\vartheta^{\hat 2}\Big).
\end{aligned}
\tag{F.1.24}
$$

Phi

Denoting $\rho^2 := (r^2+a^2)^2 - a^2\Delta\sin^2\theta$, we can introduce the vector field n of the adapted spacetime foliation by

$$
n = n^\phi\,\partial_\phi \qquad \text{with} \qquad n^\phi = -\frac{2aGmr}{c\rho^2}, \tag{F.1.25}
$$

and write the metric of spacetime in the standard form:

$$ds^2 = N^2\,dt^2 - \frac{\Sigma}{\Delta}\,dr^2 - \Sigma\,d\theta^2 - \frac{\sin^2\theta\,\rho^2}{\Sigma}\,(d\phi + n^\phi\,dt)^2. \tag{F.1.26}$$

Here $N^2 = \Delta\Sigma c^2/\rho^2$. For large distances, we find from (F.1.19), (F.1.21) and (F.1.22) the asymptotic spacetime interval

$$\begin{aligned} ds^2 = &\left[1 - \frac{2Gm}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2} + \mathcal{O}(r^{-3})\right] c^2 dt^2 \\ &- a\sin^2\theta\left[\frac{4Gm}{c^2 r} - \frac{GQ^2}{2\pi\varepsilon_0 c^4 r^2} + \mathcal{O}(r^{-3})\right] cdt\,d\phi \\ &- \left[1 + \frac{2Gm}{c^2 r} - \frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2} + \mathcal{O}(r^{-3})\right] dr^2 \\ &- r^2\left(1 + \frac{a^2}{r^2}\right)\left[d\theta^2 + \sin^2\theta d\phi^2\left(1 + \mathcal{O}(r^{-3})\right)\right]. \end{aligned} \tag{F.1.27}$$

asymKerr

When the Lie derivative of the metric vanishes, $\mathcal{L}_\xi g_{ij} = 0$, with respect to some vector field ξ, the latter is called a Killing vector of the metric. The Kerr-Newman metric possesses the two Killing vectors:

$$\overset{(t)}{\xi} = \partial_t, \qquad \text{and} \qquad \overset{(\phi)}{\xi} = \partial_\phi. \tag{F.1.28}$$

KillV

Tn GR, the knowledge of the Killing vectors provides an important information about the gravitating system. In particular, for a compact source the total mass and the total angular momentum can be given in terms of the Killing vectors by means of the so-called Komar formulas:

$$M = \frac{c}{8\pi G}\int\limits_{S_\infty} {}^*(d\overset{(t)}{k}), \qquad L = -\frac{c^3}{16\pi G}\int\limits_{S_\infty} {}^*(d\overset{(\phi)}{k}). \tag{F.1.29}$$

Komar

The integrals are taken over the spatial boundary described by the sphere of infinite radius. We used the canonical map (C.2.3) to define the 1-forms

$$\overset{(t)}{k} = \widetilde{\mathbf{g}}(\overset{(t)}{\xi}), \qquad \overset{(\phi)}{k} = \widetilde{\mathbf{g}}(\overset{(\phi)}{\xi}) \tag{F.1.30}$$

KillF

from the Killing vector fields.

It is sufficient to use the asymptotic formula (F.1.27) in (F.1.29) to prove that for the Kerr-Newman metric we have

$$M = m, \qquad L = mca. \tag{F.1.31}$$ MLa

This explains the physical meaning of the parameters m and a in the Kerr-Newman solution. It is easy to see that putting $a = 0$ brings us back to the Reissner-Nordström solution.

Electrodynamics at the outside of black holes and neutron stars

Neutron stars and black holes arise from the gravitational collaps of the ordinary matter. The gravitational effects become very strong near such objects, and GR is necessary for the description of corresponding spacetime geometry. Normally, the total electric charge of the collapsing matter is zero, and then we are left, in general, with the Kerr metric obtained from (F.1.19)-(F.1.27) by putting $Q = 0$.

Near the surface of a neutron star and outside of a black hole one can expect many interesting electrodynamical effects. To describe them, we need to solve Maxwell's equations in prescribed Kerr metric[2].

It is amazingly easy to find the exact solution of the Maxwell equations in the Kerr geometry. The crucial points are: (i) the fact that Kerr geometry describes the vacuum (matter-free) spacetime, and (ii) the existence of the two Killing vector fields (F.1.28). It is straightforward to prove that every Killing vector ξ defines a harmonic 1-form $k = \widetilde{\mathbf{g}}(\xi)$ which satifies $\Box k = 0$ and $d^{\dagger} k = 0$ in a vacuum spacetime. Recalling the Maxwell equation in the form of the wave equation (E.1.5), we immediately find that the ansatz

$$A = -\frac{B_0}{2}\left(\frac{2a}{c}\overset{(t)}{k} + \overset{(\phi)}{k}\right) \tag{F.1.32}$$ Amem1

[2]This is the main idea behind the membrane model, see Straumann [23].

yields an exact solution of the Maxwell equations on the background of the Kerr metric. Here B_0 is constant and the coefficient in the first term is chosen in accordance with (F.1.29) and (F.1.31) in order to provide a total vanishing charge. Substituting (F.1.30) into (F.1.32), we find explicitly

$$A = aB_0\,\frac{Gm}{c^2}\,(1+\cos^2\theta)\,\frac{r}{\sqrt{\Delta\Sigma}}\,\vartheta^{\hat{0}} + \frac{B_0}{2}\,(r^2+a^2)\sin^2\theta\,d\phi. \qquad \text{(F.1.33)}$$

Amem2

The physical interpretation is straghtforward: The 1-form potential (F.1.33) is a kind of superposition of the Coulomb-type electric piece [the first line, cf. (F.1.20), (F.1.23)] with the asymptotically homogeneous magnetic piece [the second line]. With respect to the coordinate foliation (for which $n = \partial_t$), the electromagnetic field strength reads $F = dA = E \wedge dt + B$ with

$$E = -\frac{B_0 GMa}{c\Sigma^2}\,[(1+\cos^2\theta)(r^2-a^2\cos^2\theta)\,dr + 2(r^2-a^2)\sin\theta\cos\theta\,d\theta], \qquad \text{(F.1.34)}$$

$$\begin{aligned} B = {} & \frac{B_0 GMa}{c^2\Sigma^2}\,[(1+\cos^2\theta)(r^2-a^2\cos^2\theta)\sin^2\theta\,dr\wedge d\phi \\ & - 2r\sin\theta\cos\theta\,(2r^2\cos^2\theta + a^2 + a^2\cos^4\theta)\,d\theta\wedge d\phi] \\ & + B_0[r\sin^2\theta\,dr\wedge d\phi + (r^2+a^2)\cos\theta\sin\theta d\theta\wedge d\phi]. \end{aligned} \qquad \text{(F.1.35)}$$

Bmem

For large distances the last line in (F.1.35) dominates yielding asymptoticaly the homogeneous constant magnetic field

$$F = B_0\,dx\wedge dy + \mathcal{O}(1/r^2). \qquad \text{(F.1.36)}$$

Here we performed the usual transformation from spherical coordinates (r,θ,ϕ) to Cartesian (x,y,z) ones. The asymptotic magnetic field is directed along the z-axis.

It is interesting to note that the electric field vanishes for the non-rotating $a = 0$ black hole. We can draw a direct parallel to the Wilson and Wilson experiment where the magnetic field induced the electric field inside a rotating body. The spacetime of a rotating Kerr geometry acts similarly and induces the electric field around the black hole.

In the membrane approach[3], the physics outside a rotating black hole is described with the help of a model when a horizon is treated as a conducting membrane which possesses certain surface charge and current density, as well as the surface resistivity. One can develop, in particular, the mechanism of extracting the (rotational) energy from a black hole by means of the external magnetic fields.

Force-free fields

Near a black hole or a neutron star the force-free fields can naturally emerge in the plasma of electrons and positrons. In Sec. B.2.2, we have defined such electromagnetic fields by the condition of vanishing of the Lorentz force (B.2.12). Using the spacetime relation (F.1.4), we can now develop a more substantial analysis of the situation. The force-free condition now reads:

$$(e_\alpha \rfloor F) \wedge d^*F = 0\,. \qquad \text{(F.1.37)}$$ FFree

To begin with, let us recall the *sourceless* solution of above. The magnetic field (F.1.35) has the evident structure:

$$B = d\Psi \wedge d\phi, \qquad \text{(F.1.38)}$$

with

$$\Psi = B_0 \sin\theta \left(\frac{r^2+a^2}{2} - \frac{GMa^2r}{c^2\Sigma}(1+\cos^2\theta)\right). \qquad \text{(F.1.39)}$$

Thus, the magnetic field is manifestly axially symmetric, that is, its Lie derivative with respect to the vector field ∂_ϕ is zero:

$$\mathcal{L}_{\partial_\phi} B = d(\partial_\phi \rfloor B) + \partial_\phi \rfloor dB = -\,dd\Psi \equiv 0. \qquad \text{(F.1.40)}$$ LieB

Here we used the Maxwell equation $dB = 0$ which are identically fulfilled for any function Ψ which does not depend on ϕ, $\Psi = \Psi(r,\theta)$.

[3] See, e.g., Straumann [23] and the literature therein.

Returning to the problem under consideration (i.e., *with* nontrivial plasma source), we will also demand that the magnetic field be axially symmetric. As more general structure of the field we then expect

$$B = d\Psi \wedge d\phi + \Phi \, dr \wedge d\theta. \tag{F.1.41}$$

Bfree

Clearly, this 2-form also satisfies the axial symmetry condition (F.1.40) for every Φ. The Maxwell equation $dB = 0$ is again fulfilled provided $\Phi = \Phi(r, \theta)$. Moreover, usually one assumes that $\Phi = \Phi(\Psi)$.

Now we need also the ansatz for the electric 1-form E. Taking into account the axial symmetry, a natural assumption reads

$$E = v \rfloor B, \tag{F.1.42}$$

Efree1

where the vector field

$$v = \Omega \, \partial_\phi \tag{F.1.43}$$

can be interpreted as the rotational velocity of the magnetic field lines. The function Ω does not depend on the angular coordinate ϕ. Furthermore, substituing (F.1.41) into (F.1.42), we find

$$E = -\,\Omega \, d\Psi. \tag{F.1.44}$$

Efree2

The Maxwell equation $dE = 0$ is fulfilled if $\Omega = \Omega(\Psi)$.

Combining (F.1.41) and (F.1.44), we find the general ansatz for the electromagnetic field strength 2-form:

$$F = -\,\Omega \, d\Psi \wedge dt + d\Psi \wedge d\phi + \Phi \, dr \wedge d\theta. \tag{F.1.45}$$

FBEfree

This must be inserted into the force-free condition (F.1.37) in which we will use the natural coordinate frame, $e_\alpha = \delta^i_\alpha \partial_i$. Direct inspection shows that equation (F.1.37) is identically fulfilled for ∂_t and ∂_ϕ. Substituting (F.1.45) into (F.1.37) for ∂_r and ∂_θ yields a nontrivial differential equation:

$$d(\alpha \, {}^*d\Psi) + \beta \, d\Psi \wedge {}^*d\Psi + \gamma \, {}^*\Phi \frac{d\Phi}{d\Psi} = 0. \tag{F.1.46}$$

Pfree

Here the functions $\alpha = \alpha(r,\theta)$, $\beta = \beta(r,\theta)$, $\gamma = \gamma(r,\theta)$ are constructed from Ω, its derivative, and from the components of the spacetime metric. Equation (F.1.46) is called Grad-Shafranov equation and, with the given functions $\Omega = \Omega(\Psi)$ and $\Phi = \Phi(\Psi)$, the solution Ψ of (F.1.46) completely describes the force-free electromagnetic field configuration. The corresponding distribution of the charge and current density is derived from the Maxwell equation $J = dH$.

F.1.2 Classical (1st quantized) Dirac field

In classical electrodynamics, the electric current 3-form J is phenomenologically specified, it cannot be resolved any further. We know, however, that electric charge is carried by the fundamental particles, namely the leptons and the quarks. For many everyday effects, the electron and the proton (consisting of 3 confined quarks) are responsible. The 2nd quantized Dirac theory governs the behavior of the *electron*.

If the energies involved in an experiment, are not too high (compared to the mass of the electron), then the electron can be approximately viewed as a classical matter wave, i.e., only a 1st quantized Dirac wave function. An electron microscope and its resolution may well be described in such a manner; similarly, an electron interferometer for sensing rotation (Sagnac type of effect) needs no more refined description. Let us then assume that the Dirac matrices, referred to an orthonormal coframe, are given by

$$\gamma_{(\alpha}\,\gamma_{\beta)} = o_{\alpha\beta}\,. \qquad \text{(F.1.47)}$$ Diracmat

Then we can introduce Dirac-algebra valued 1-forms $\gamma := \gamma_\alpha\,\vartheta^\alpha$. The Dirac equation then reads

$$i\hbar^*\gamma \wedge D\,\Psi + {}^*mc\,\Psi = 0\,, \quad \text{with} \qquad D = d + i\frac{e}{\hbar}A\,. \quad \text{(F.1.48)}$$ Dirac

The Dirac adjoint is $\overline{\Psi} := \gamma^0\Psi^\dagger$. The Dirac equation can be derived from a Lagrangian $L_{\rm D}$. The conserved electric current turns out to be

$$J := \frac{\delta L_{\rm D}}{\delta A} = -\, ie\, \overline{\Psi}\tau^{*}\gamma\Psi\,, \qquad dJ = 0\,. \tag{F.1.49}$$ Diraccurrent

The energy-momentum current of the Dirac field, according to the Lagrange-Noether procedure developed in Sec.B.5.5, reads

$$\Sigma_\alpha = \frac{\delta L_{\rm D}}{\delta \vartheta^\alpha} = \frac{i\hbar}{2}\left(\overline{\Psi}\,{}^{*}\gamma D_\alpha \Psi - D_\alpha \overline{\Psi}\,{}^{*}\gamma\Psi\right)\,, \qquad D\Sigma_\alpha = 0\,. \tag{F.1.50}$$ sigmaDi

Here $D_\alpha := e_\alpha \rfloor D$. Therefore the self-consistent Dirac-Maxwell system reads

$$d^{\star}F = J\,, \qquad dF = 0 \tag{F.1.51}$$ DiMax1

$$J = -\, ie\, \overline{\Psi}\tau^{*}\gamma\Psi\,, \qquad i\hbar^{*}\gamma \wedge D\,\Psi + {}^{*}mc\,\Psi = 0\,. \tag{F.1.52}$$ DiMax2

With gravity, we have to generalize the spinor covariant derivative which now should read

$$D = d + i\frac{e}{\hbar}A + \frac{i}{4}\,\widehat{\sigma}_{\alpha\beta}\,\Gamma^{\alpha\beta}, \tag{F.1.53}$$

where, as usual, $\widehat{\sigma}^{\alpha\beta} := i\gamma^{[\alpha}\gamma^{\beta]}$. In order to write the Einstein gravitational field equation, have to symmetrize the energy-momentum current,

$$\sigma_\alpha = \Sigma_\alpha - D\mu_\alpha \qquad \text{with} \qquad \tau_{\alpha\beta} = \vartheta_{[\alpha} \wedge \mu_{\beta]}\,, \tag{F.1.54}$$ Disym

where $\tau_{\alpha\beta} = (\partial L_{\rm D}/D\Psi)\,(i\widehat{\sigma}_{\alpha\beta}/4)\,\Psi$ is the canonical spin current 3-form of the Dirac field. Then the complete Einstein-Dirac-Maxwell system, together with (E.1.23,F.1.51,F.1.52), reads

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma} \wedge \widetilde{R}^{\beta\gamma} = \frac{8\pi G}{c^3}\left(\overset{\rm Max}{\Sigma}{}_{\alpha} + \overset{\rm D}{\sigma}_{\alpha}\right)\,, \tag{F.1.55}$$

Alternative Dirac coupling to gravity via the Einstein-Cartan theory. We allow for a metric compatible connection $\Gamma^{\alpha\beta} = \Gamma^{\beta\alpha}$ carrying a torsion piece. Then the spin current of the Dirac field can be defined according to

$$\overset{\rm D}{\tau}_{\alpha\beta} := -\frac{\delta \overset{\rm D}{L}}{\delta\Gamma^{\alpha\beta}} = \frac{\hbar}{4}\,\vartheta_\alpha \wedge \vartheta_\beta \wedge \overline{\Psi}\gamma\gamma_5\Psi.\,, \qquad D\,\overset{\rm D}{\tau}_{\alpha\beta} + \vartheta_{[\alpha} \wedge \overset{\rm D}{\Sigma}_{\beta]} = 0\,. \tag{F.1.56}$$ tauDyn

Then, for the gravitational sector of the the Einstein-Cartan-Dirac-Maxwell system, we find

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\wedge R^{\beta\gamma}=\frac{8\pi G}{c^3}\,\left(\overset{\text{Max}}{\Sigma}{}_{\alpha}+\overset{\text{D}}{\Sigma}_{\alpha}\right)\,, \tag{F.1.57}$$

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\wedge T^{\gamma}=\frac{8\pi G}{c^3}\,\overset{\text{D}}{\tau}_{\alpha\beta}\;. \tag{F.1.58}$$

This is a viable alternative to the Einstein-Dirac-Maxwell system. It looks very symmetric. Note that the electromagnetic field doesn't carry dynamical spin.

If we take either the Einstein-Dirac-Maxwell or Einstein-Cartan-Dirac-Maxwell system, we can in any case also study the gravitational properties of the Dirac electron, see [10].

Remark on superconductivity quasi-classically understood: Ginzburg-Landau theory

Generalizing the classical Maxwell-London theory of superconductivity, GL achieved, by introducing a complex 'order parameter', a quasi-classical description of superconductivity for $T = 0$. One consequence of this theory is the Abrikosov lattice of magnetic flux lines, see Sec.B.3.1. The Lagragian...

F.1.3 Topology and electrodynamics

In our book we in fact did not discuss genuine topological aspects of electrodynamics. However, topology can play a very important and nontrivial role in electrodynamics and in magnetohydrodynamics[4].

A word of caution is in order: One should carefully distinguish the physical situations in which an underlying spacetime (or space) has a complicated topology from the case when the electromagnetic *field configuration* is topologically nontrivial. Usually the decisive role is played by the *pure gauge* contribution to the electromagnetic potential 1-form.

A manifest example is given by the force-free magnetic fields which approximately describe the twisted flux tubes in the models of solar prominence (sheets of luminous gas emanating from

[4]See the reviews of Moffatt and Marsh [19, 17, 18].

the sun's surface)[5]. Recall the equation (B.2.14) which determines the force-free magnetic field. It is identically fulfilled when

$$\underline{d}\mathcal{H} = \alpha\, B. \tag{F.1.59}$$

dHaB

Indeed, since B is a transversal 2-form (i.e., living in 3 spatial dimensions), we have $B \wedge B \equiv 0$. Consequently, $B \wedge e_a \rfloor B = 0$. Thus (F.1.59) solves (B.2.14) for any coefficient function $\alpha = \alpha(x)$.

Note that the ansatz (F.1.59) can be used even in the metric-free formulation of electrodynamics. In Maxwell-Lorentz electrodynamics, (F.1.59) further reduces to

$$\underline{d}\,{}^{\underline{*}}B = \alpha\, B. \tag{F.1.60}$$

dBaB

It is worthwhile to note that this is the field equation of the so-called "topologically massive electrodynamics" in 3 dimensions[6] provided α is constant.

Specializing to the axially symmetric configurations in Minkowski spacetime, we can easily find a solution of (F.1.60) for any choice of α. For example, in cylindrical coordinates (ρ, ϕ, z), for $\alpha(\rho) = 2/[a(1+\rho^2/a^2)]$, we obtain

$$B = \frac{B_0\,\rho}{1+\frac{\rho^2}{a^2}}\, d\rho \wedge \left(d\phi - \frac{1}{a}\,dz\right). \tag{F.1.61}$$

Btwist

Here a is a constant parameter which determines the twist. This solution describes a uniformly twisted flux tube. Although $\underline{d}\alpha \neq 0$, nevertheless $\underline{d}\alpha \wedge B = 0$ which provides the consistency of the solution.

It is straightforward to read off from (F.1.61) the potential 1-form,

$$\mathcal{A} = \frac{B_0 a^2}{2} \log\left(1 + \frac{\rho^2}{a^2}\right)\left(d\phi - \frac{1}{a}\,dz\right) + d\chi, \tag{F.1.62}$$

Atwist

where χ is an arbitrary gauge function.

---

[5]This is discussed in Marsh [18], e.g.

[6]See Deser et al. [4].

There exist various topological numbers (or invariants) which evaluate the topological complexity of an electric and magnetic field configuration. The so-called magnetic helicity provides an explicit example of such a number. Defined by the integral

$$h := \int_V \mathcal{A} \wedge B, \tag{F.1.63}$$

helicity

the helicity measures the "linkage" of the magnetic field lines. One can establish a direct relation of h to the classical Hopf invariant, which classifies the maps of a 3-sphere on a 2-sphere, and can interpret it in terms of the Gauss linking number. It is instructive to compare (F.1.63) with (B.3.16) and (B.3.17).

Turning again to the twisted flux tube solution above, we see form (F.1.62) and (F.1.61) that in the exterior product of $\mathcal{A}$ with B only the contribution from the pure gauge survives: $\mathcal{A} \wedge B = \underline{d}\chi \wedge B = \underline{d}(\chi B)$. As a result, a nonrivial value of the helicity (F.1.63) can only be obtained by assuming a toroidal topology of space. This can be achieved by gluing two two-dimensional cross-sections at some values z_1 and z_2 of the third coordinate and, moreover, by assigning a nontrivial jump $\delta\chi = \chi(z_2) - \chi(z_1)$ to the gauge function.

There is a close relation between magnetic interaction energy and helicity. For the *force-free* magnetic field, we have explicitly,

$$E_{\text{mag}} = \int \mathcal{A} \wedge j = \int \alpha \mathcal{A} \wedge B, \tag{F.1.64}$$

where we used the Oersted-Ampère law $j = \underline{d}\mathcal{H}$ and the ansatz (F.1.59). When α is constant, the energy is proportional to the magnetic helicity.

More on the electrodynamics in multi-connected domains can be found in Marsh [17, 18]. The corresponding effects underline the physical importance of the electromagnetic potential 1-form A. Another manifestation of topology in electrodynamical systems is of a more quantum nature: the Aharonov-Bohm effect[7].

[7] See the theoretical discussion by Aharonov and Bohm [1] and the first experimental findings by Chambers [3], a recent evaluation has been given by Nambu [20].

Remarks on plasma physics and magnetohydrodynamics

As we saw already above, when we discussed solar prominence, topological effect of mainly magnetic configurations play a role in plasma physics in general and in magnetohydrodynamics specifically, see Cap [2], Hora [11], and also Knoepfel [14].

F.1.4 Remark on possible violations of Poincaré invariance

The first four axioms are completely free of the metric concept. The Poincaré group doesn't play a role at all. The conformal group is brought in by the fifth axiom. In fact, we start from a *linear* ansatz and end up, via some 'technical assumptions', at the Maxwell-Lorentz spacetime relation. Can we modify the fifth axiom in a suitable way? Optical properties of the cosmos near the big bang. Is there optical activity, birefringence etc. of the vacuum? If yes, then the fifth axiom has to go willy nilly. See Kostelecky [15] and references given there.

F.2

Quantum physics *(preliminary)*

F.2.1 QED

Our approach is essentially classical. This is a legitimate assumption within the well known and rather wide limits of the idealized representation of the electric charges, currents by means of particles (and continuous media) and electromagnetic radiation by means of classical waves of electric and magnetic fields.

However it is experimentally well established that an electron under certain conditions may display the wave properties, whereas the electromagnetic radiation may sometimes be treated in terms of particles, i.e. photons. Quantum electrodynamics (QED) takes into account these facts.

The mathematical framework of QED is the scheme of the second quantization in which the electromagnetic (Maxwell) and electron (Dirac spinor) fields are replaced by the field operators acting in the Hilbert space of quantum states. Physical processes are then described in terms of creation, annihilation, and propagation of quanta of electromagnetic and spinor fields. The photon γ is massless and has spin 1, whereas electron is massive and has spin 1/2.

In QED, the electromagnetic 1-form potential A plays a fundamental role, representing the "generalized coordinate" with the excitation H being its canonically conjugated "momentum". Also the gauge symmetry $A \to A + d\chi$ moves to the center of the theory: One can consistently build the electrodynamics on the basis of the local gauge invariance principle. The underlying symmetry group is the Abelian $U(1)$ and A naturally arizes as the $U(1)$-gauge field potential which is geometrically interpreted as the connection in the principal $U(1)$-bundle of the spacetime. The gauge freedom is responsible for the masslessness of a photon, and thus ultimately for the long range character of electromagnetic interaction.

QED correctly describes many quantum phenomena involving electrons and photons. It is very well experimentally verified, in fact this is the most precisely tested physical theory. The most famous and precise experiments are the proof of the predicted value of the anomalous magnetic moment of the electron and the observation of the shift of energy levels in atoms.

The success of QED is to a great extent related to the smallness of the coupling constant $\alpha_{\rm f} = \frac{e^2}{4\pi\varepsilon_0\hbar c}$ of the theory which makes it possible to effectively use the perturbation approach. At the same time, QED is not free of deficiencies. The most serious is the problem of divergences and the need of the regularization and renormalization methods.

F.2.2 Electro-weak unification

Recently, the considerable progress has been achieved in the construction of unified theories of the physical interactions. In particular, QED is found to be naturally unified with the weak interaction.

A typical example of a quantum process governed by the weak interaction is the decay of a neutron into the proton, electron and antineutrino. The modern understanding of the weak forces is based on the gauge approach. In the Weinberg-Salam model, the fundamental symmetry group $SU(2) \times U(1)$ gives rise to the four gauge fields as the mediators of the electro-weak interaction: charged W^+ and W^- vector bosons, a neutral Z^0 intermediate vector boson, and the photon γ. The spin 1 particles $W^\pm, Z^0$ become massive via the mechanism of the spontaneous symmetry breaking of the gauge group, whereas γ corresponds to the unbroken exact subgroup and remains massless.

In the so called standard model, the group $SU(2) \times U(1)$ is enlarged to $SU(3) \times SU(2) \times U(1)$ and the resulting theory represents the unification of electromagnetic, weak and strong forces. The color group $SU(3)$ brings to life the 8 additional gauge field 1-forms A^a, $a = 1, \ldots, 8$ which are called the gluon potentials. The standard model is responsible for the description of interaction of quarks (fermionic constituents of the barions) and leptons (electron, muon, tau, and neutrinos) by means of the gluons A^a and the intermediate gauge bosons $W^\pm, Z^0$ and γ.

Of the most ambitious development of the unification program it is worthwhile to mention the (super)string model which aims ultimately in constructing the consitent quantum gravity theory. In the string model the point particles are replaced by the extended fundamental objects. The main advantage of such an approach is the possibility of elimination of the quantum divergences.[1]

What can we learn from our approach for the quantum field theory? Perhaps that the Abelian *axion* is a very natural candidate for a particle, in contrast to the Dirac monopole. Also our approach could give hints how a possible violation of Poincaré invariance could be brought about. Will the topological results, see above, turn out to be important in higer-dimensional theories?

[1]There exist a lot of texts on the string theory. As a good introduction, one may consult [21].

F.2.3 Quantum Chern-Simons and the QHE

In Sec. B.4.4 we have presented a classical description of the quantum Hall effect. Such an approach is however only qualitatively correct. The true understanding of the QHE is possible when the quantum aspects of the phenomenon are carefully studied.

We have no tools in our book to pursue this goal. The best thing we can do is to address the interested reader to the corresponding literature[2]. The QHE is described within the framework of the effective topological quantum field theory with a Chern-Simons action. The explanation of the quantization of the Hall conductance σ_{H} is the ultimate goal achieved in these studies.

Incidentally, if one describes the quantum Hall effect for low lying Landau levels, then the concept of a *composite fermion* is very helpful: it consists of one electron and an even number of fluxoids is attached to it.[3] Isn't that a very clear adiitional indication of what the fundamental quantities are in electrodynamics? Namely, electric charge (see first axiom) and magnetic flux (see third axiom).

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[2] See Fröhlich and Pedrini [7], e.g..

[3] See Jain [12, 13] and, in this general context also Nambu [20].

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